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Superstatistics

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Abstract

We consider nonequilibrium systems with complex dynamics in stationary states with large fluctuations of intensive quantities (e.g. the temperature, chemical potential or energy dissipation) on long time scales. Depending on the statistical properties of the fluctuations, we obtain different effective statistical mechanical descriptions. Tsallis statistics follows from a χ^2-distribution of an intensive variable, but other classes of generalized statistics are obtained as well. We show that for small variance of the fluctuations all these different statistics behave in a universal way.

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Einstein never accepted Boltzmann's principle $S=k \log W$, because he argued that the statistics (W) of a system should follow from its dynamics and, in principle, could not be postulated a priori [1,2]. Remarkably, the Boltzmann–Gibbs (BG) statistics works perfectly for classical systems with short range forces and relatively simple dynamics in equilibrium. A possible consequence of Einstein's criticism is that for systems with sufficiently complex dynamics, other than BG statistics might be appropriate. Such a statistics has indeed been proposed by Tsallis [3] and has been observed in various complex systems [4–17]. For these types of systems the formalism of nonextensive statistical mechanics [18,19] is a useful theoretical concept, in the sense that a more general Boltzmann factor is introduced which depends on an entropic index q and

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which, for $q = 1$, reduces to the ordinary Boltzmann factor. Applications of the formalism have been mainly reported for multifractal systems, systems with long-range interactions and nonequilibrium systems with a stationary state. Several types of generalized stochastic dynamics have been recently constructed for which Tsallis statistics can be proved rigorously [4–6]. Physical applications include 2-d and 3-d turbulence [7–14], momentum spectra of hadronic particles produced in e^+e^- annihilation experiments [15,16], the statistics of cosmic rays [17], and many other phenomena.

In this paper we will quite generally deal with nonequilibrium systems with a long-term stationary state that possess a spatio-temporally fluctuating intensive quantity. We will show that after averaging over the fluctuations one can obtain as an effective description not only Tsallis statistics but an infinite set of more general statistics, which we will call 'superstatistics'. This name is chosen because the new statistics represents a kind of 'statistics of statistics'. We will consider several concrete examples. A parameter q can be defined for all these new statistics and given a physical interpretation. For $q = 1$ all superstatistics reduce to the Boltzmann factor $e^{-\beta_0 E}$ of ordinary statistical mechanics. One of our main results is universality for small amplitudes of the fluctuations. If the variance of the fluctuations is small, then the first-order corrections to the ordinary Boltzmann factor are the same for all superstatistics (i.e., those of Tsallis statistics). On the other hand, for large variances quite different behaviour from Tsallis statistics can be generated. We calculate the higher-order correction terms of various examples of superstatistics.

Experimental evidence for superstatistics that is nearly Tsallis but has tiny corrections beyond Tsallis has very recently been provided by Jung and Swinney [23], by careful analysis of velocity fluctuations in a turbulent Taylor–Couette flow. Moreover, experimental studies from fusion plasma physics provide possible evidence for non-Tsallis superstatistics based on F-distributions [24]. Sattin and Salasnich actually consider a 2-parameter generalization of Tsallis statistics in their paper [24] and point out the usefulness of further generalizations. All this illustrates the need for a more general approach towards generalized statistics for complex dynamical systems.

Let us now outline the theory. Consider a driven nonequilibrium system that is composed of regions that exhibit spatio-temporal fluctuations of an intensive quantity. This could be the inverse temperature β, or equally well the pressure, chemical potential, or the energy dissipation rate in a turbulent fluid [5,11,12]. We will select the temperature here as our fluctuating quantity but it could be anything of the above. We note that even the fluctuation theory of Onsager and Machlup does not treat this case and is restricted to fluctuations of extensive thermodynamic variables [20]. We consider a nonequilibrium steady state of a macroscopic system, made up of many smaller cells that are temporarily in local equilibrium. Within each cell, β is approximately constant. Each cell is large enough to obey statistical mechanics, but has a different β assigned to it, according to a probability density $f(\beta)$. We assume that the local temperatures in the various cells change on a long time scale T. This time scale T is much larger than the relaxation time that the single cells need to reach local equilibrium. It is clear that our model is a suitable approximation for a continuously varying temperature field that has spatial correlation length of the order of the cell size L and a temporal correlation length T.

A Brownian test particle moves for a while in a certain cell with a given temperature, then moves to the next cell, and so on. Its velocity v obeys

$$\dot{v} = -\gamma v + \hat{\sigma} L(t) , \tag{1}$$

where $L(t)$ is Gaussian white noise. The inverse temperature of each cell is related to the parameters γ and $\hat{\sigma}$ by $\beta = \gamma/\hat{\sigma}^2$. However, unlike ordinary Brownian motion, the parameter β is not constant but changes temporally on the time scale T and spatially on the scale L. These changes are ultimately produced by the very complex dynamics of the environment of the Brownian particle. It has been proved by one of us [5] that after averaging over the fluctuating β this simple generalized Langevin model generates Tsallis statistics for v if β is a χ^2-distributed random variable. Moreover, the obtained distributions for v were shown to fit quite precisely distributions of longitudinal velocity differences as measured in turbulent Taylor–Couette flows [5,10], as well as measurements in Lagrangian turbulence [12,13].

Let us here generalize this approach to general distributions $f(\beta)$ and general (effective) Hamiltonians. In the long-term run ($t \gg T$), the stationary probability density of our nonequilibrium system arises out of Boltzmann factors $e^{-\beta E}$ associated with the cells that are averaged over the various fluctuating inverse temperatures β. If E is the energy of a microstate associated with each cell, we may write

$$B(E) = \int_0^\infty d\beta f(\beta) \, e^{-\beta E} , \tag{2}$$

where B is a kind of effective Boltzmann factor for our nonequilibrium system, the superstatistics of the system. In a sense, it represents the statistics of the statistics ($e^{-\beta E}$) of the cells of the system. $B(E)$ may significantly differ from the ordinary Boltzmann factor, which is recovered for $f(\beta) = \delta(\beta - \beta_0)$. For the above simple example of a Brownian test particle of mass 1 one has $E = \frac{1}{2}v^2$, so that the long-term stationary state consists of a superposition of Gaussian distributions $e^{-\beta E}$ that are weighted with the probability density $f(\beta)$ to observe a certain β. But our consideration in the following applies to arbitrary energies E associated with the cells, not only $E = \frac{1}{2}v^2$. The central hypothesis of our paper is that generalized Boltzmann factors of the form (2) are physically relevant for large classes of dynamically complex systems with fluctuations.

One immediately recognizes that the generalized Boltzmann factor of superstatistics is given by the Laplace transform of the probability density $f(\beta)$. Although there are infinitely many possibilities, certain criteria must be fulfilled which significantly reduce the number of physically relevant cases:

1. $f(\beta)$ cannot be any function but must be a normalized probability density. It may, in fact, be a physically relevant density from statistics, say Gaussian, uniform, chi-squared, lognormal, but also other, as yet unidentified, distributions could be considered if the underlying dynamics is sufficiently complex.
2. The new statistics must be normalizable, i.e., the integral $\int_0^\infty B(E)\, dE$ must exist, or in general the integral $\int_0^\infty \rho(E) B(E)\, dE$, where $\rho(E)$ is the density of states.
3. The new statistics should reduce to BG-statistics if there are no fluctuations of intensive quantities at all.

We will now consider several examples of superstatistics (later we will see what is universal to all of them). The distribution $f(\beta)$ is ultimately determined by the invariant measure of the underlying dynamical system and a priori unknown. Gaussian β-fluctuations would generate one of the simplest $f(\beta)$, but these types of models are unphysical since they allow for negative β with some nonzero probability. Instead, we need distributions where the random variable β is always positive.

Uniform distribution: As a very simple model where everything can be calculated analytically let us first consider a uniform distribution of β on some interval $[a, a+b]$, i.e.,

$$f(\beta) = \frac{1}{b} \tag{3}$$

for $0 \leqslant a \leqslant \beta \leqslant a+b$, whereas $f(\beta) = 0$ elsewhere. The mean of this distribution is

$$\beta_0 = a + \frac{b}{2} \tag{4}$$

and for the variance one obtains $\sigma^2 = \langle \beta^2 \rangle - \beta_0^2 = b^2/12$. The superstatistics of this model follows from the generalized Boltzmann factor

$$B = \int_0^\infty \mathrm{e}^{-\beta E} f(\beta)\,\mathrm{d}\beta = \frac{1}{bE}\left(\mathrm{e}^{-(\beta_0 - (1/2)b)E} - \mathrm{e}^{-(\beta_0 + (1/2)b)E}\right) . \tag{5}$$

This is normalizable for any $E \geqslant 0$. Hence this is a reasonable model of superstatistics. For small bE one obtains, by series expansion of $\mathrm{e}^{-(1/2)bE}$,

$$B = \mathrm{e}^{-\beta_0 E}(1 + \tfrac{1}{24} b^2 E^2 + \tfrac{1}{1920} b^4 E^4 + \cdots) . \tag{6}$$

Clearly, for $b \to 0$ ordinary statistical mechanics is recovered.

2-*level distribution*: Suppose the subsystems can switch between two different discrete values of the temperature, each with equal probability. A physical example might be a Brownian particle that can switch between two different states with different friction constants, or two different chemical potentials. The probability density is given by

$$f(\beta) = \tfrac{1}{2}\delta(a) + \tfrac{1}{2}\delta(a+b) . \tag{7}$$

The average of β is again given by

$$\beta_0 = a + \frac{b}{2} \tag{8}$$

and for the variance one obtains $\sigma^2 = \langle \beta^2 \rangle - \beta_0^2 = b^2/4$. The generalized Boltzmann factor becomes

$$B = \int_0^\infty \mathrm{e}^{-\beta E} f(\beta)\,\mathrm{d}\beta = \mathrm{e}^{-\beta_0 E}\,\frac{1}{2}\,(\mathrm{e}^{(1/2)bE} + \mathrm{e}^{-(1/2)bE}) . \tag{9}$$

This is normalizable for any $E \geqslant 0$. For small bE one obtains

$$B = \mathrm{e}^{-\beta_0 E}(1 + \tfrac{1}{8} b^2 E^2 + \tfrac{1}{384} b^4 E^4 + \cdots) . \tag{10}$$

Gamma-distribution: The assumption of a Gamma (or χ^2-) distributed inverse temperature β leads to Tsallis statistics, so far the most relevant example of superstatistics.

We may write the Gamma distribution as

$$f(\beta) = \frac{1}{b\Gamma(c)} \left(\frac{\beta}{b}\right)^{c-1} \mathrm{e}^{-\beta/b} , \tag{11}$$

where $c > 0$ and $b > 0$ are parameters. The number $2c$ can be interpreted as the effective number of degrees of freedom contributing to the fluctuating β. The average of β is $\beta_0 = \int_0^\infty \beta f(\beta)\,\mathrm{d}\beta = bc$ and for the variance of β one obtains $\sigma^2 = b^2 c$.

The Gamma distribution naturally arises if $n = 2c$ independent Gaussian random variables X_k with average 0 are squared and added. If we write

$$\beta = \sum_{k=1}^{n} X_k^2 \tag{12}$$

then this β is Gamma distributed with density (11). The average of β is of course given by n times the variance of the Gaussian random variables X_k. In this sense the Gamma distribution arises very naturally for a fluctuating environment with a finite number n of degrees of freedom.

The integration over β yields the generalized Boltzmann factor

$$B = \int_0^\infty \mathrm{e}^{-\beta E} f(\beta)\,\mathrm{d}\beta = (1 + bE)^{-c} , \tag{13}$$

i.e., the generalized Boltzmann factor $(1+(q-1)\beta_0 E)^{-1/(q-1)}$ of nonextensive statistical mechanics [3,18,19] if we identify $c = 1/(q-1)$ and $bc = \beta_0$. It is normalizable for $c > 1$ (if $\rho(E) = 1$).

We may also write $B = \exp\{-c \log(1 + bE)\}$ and expand the logarithm for small bE. The result can be written in the form

$$B = \mathrm{e}^{-\beta_0 E}(1 + \tfrac{1}{2}(q-1)\beta_0^2 E^2 - \tfrac{1}{3}(q-1)^2 \beta_0^3 E^3 + \cdots) , \tag{14}$$

where q is the entropic index of nonextensive statistical mechanics.

Log-normal distribution: The log-normal distribution

$$f(\beta) = \frac{1}{\beta s \sqrt{2\pi}} \exp\left\{ \frac{-(\log(\beta/m))^2}{2s^2} \right\} \tag{15}$$

yields yet another possible superstatistics (see also Ref. [21] for a related turbulence model); m and s are parameters. The average β_0 of the above log-normal distribution is given by $\beta_0 = m\sqrt{w}$ and the variance by $\sigma^2 = m^2 w(w-1)$, where $w := \mathrm{e}^{s^2}$. The generalized Boltzmann factor $B = \int_0^\infty f(\beta)\mathrm{e}^{-\beta E}\,\mathrm{d}\beta$ cannot be evaluated in closed form, but in leading order we obtain for small variance of the inverse temperature fluctuations

$$B = \mathrm{e}^{-\beta_0 E} \left(1 + \tfrac{1}{2} m^2 w(w-1)E^2 - \tfrac{1}{6} m^3 w^{3/2}(w^3 - 3w + 2)E^3 + \cdots\right) . \tag{16}$$

F-distribution:

The last example we want to consider is that of a $\beta \in [0, \infty]$ distributed according to the F-distribution [22]

$$f(\beta) = \frac{\Gamma((v+w)/2)}{\Gamma(v/2)\Gamma(w/2)} \left(\frac{bv}{w}\right)^{v/2} \frac{\beta^{(v/2)-1}}{(1 + (vb/w)\beta)^{(v+w)/2}} . \tag{17}$$

Here w and v are positive integers and $b>0$ is a parameter. We note that for $v=2$ we obtain a Tsallis distribution. However, this is a Tsallis distribution in β-space, not in E-space as in Eq. (13).

The average of β is given by

$$\beta_0 = \frac{w}{b(w-2)} \tag{18}$$

and the variance by

$$\sigma^2 = \frac{2w^2(v+w-2)}{b^2v(w-2)^2(w-4)} . \tag{19}$$

The Laplace transform cannot be obtained in closed form, but for small variance of the fluctuations we obtain the series expansion

$$B(E) = \mathrm{e}^{-\beta_0 E}\left(1 + \frac{w^2(v+w-2)}{b^2v(w-2)^2(w-4)}E^2 - \frac{4w^3(v+w-2)(2v+w-2)}{3b^3v^{3/2}(w-2)^3(w-4)(w-6)}E^3 + \cdots\right) . \tag{20}$$

F-distribution in β-space have also been studied by Sattin and Salasnich [24], who point out possible applications in fusion plasma physics.

While in general the large-E behavior is different for all superstatistics (it strongly depends on the function $f(\beta)$), we now show that the low-E behavior is universal. We note that for the above five distribution functions the first-order corrections to the Boltzmann factor $\mathrm{e}^{-\beta_0 E}$ in Eqs. (6), (10), (14), (16) and (20) can all be written in a universal form. When expressing the generalized Boltzmann factor in terms of the average temperature β_0 and the variance σ^2 of the various probability densities $f(\beta)$, one always obtains the same result for small σ,

$$B = \mathrm{e}^{-\beta_0 E}(1 + \tfrac{1}{2}\sigma^2E^2 + O(\sigma^3E^3)) , \tag{21}$$

where σ is the standard deviation of the distribution $f(\beta)$. Thus in the limit of approaching ordinary statistics, the generalized Boltzmann factor of superstatistics has a universal quadratic correction term $(1+\frac{1}{2}\sigma^2E^2)$.

Let us now prove this universality. For any distribution $f(\beta)$ with average $\beta_0 := \langle\beta\rangle$ and variance $\sigma^2 := \langle\beta^2\rangle - \beta_0^2$ we can write

$$\begin{aligned} B &= \langle \mathrm{e}^{-\beta E}\rangle = \mathrm{e}^{-\beta_0 E}\mathrm{e}^{+\beta_0 E}\langle \mathrm{e}^{-\beta E}\rangle \\ &= \mathrm{e}^{-\beta_0 E}\langle \mathrm{e}^{-(\beta-\beta_0)E}\rangle \\ &= \mathrm{e}^{-\beta_0 E}\left(1 + \frac{1}{2}\sigma^2E^2 + \sum_{r=3}^{\infty}\frac{(-1)^r}{r!}\langle(\beta-\beta_0)^r\rangle E^r\right) . \end{aligned} \tag{22}$$

Here the coefficients of the powers E^r are the rth moments of the distribution $f(\beta)$ about the mean, which can be expressed in terms of the ordinary moments as

$$\langle(\beta-\beta_0)^r\rangle = \sum_{j=0}^{r}\binom{r}{j}\langle\beta^j\rangle(-\beta_0)^{r-j} . \tag{23}$$

Due to the universality proved above, it now makes sense to define a universal parameter q for any superstatistics, not only for Tsallis statistics. By comparing Eq. (14) and (21), we generally define a parameter q by the relation

$$(q-1)\beta_0^2=\sigma^2 \tag{24}$$

or equivalently

$$q=\frac{\langle\beta^2\rangle}{\langle\beta\rangle^2}\ . \tag{25}$$

For example, for the log-normal distribution we obtain from Eq. (25) $q=w$ and for the F-distribution $q=1+2(v+w-2)/v(w-4)$. The physical meaning of this generally defined parameter q is that $\sqrt{q-1}=\sigma/\beta_0$ is just the coefficient of variation of the distribution $f(\beta)$, defined by the ratio of standard deviation and mean. If there are no fluctuations of β at all, we obtain $q=1$ as required. For small σE the corrections to the ordinary Boltzmann factor $e^{-\beta_0 E}$ are universal and given by the factor $(1+(q-1)\beta_0^2E^2/2)$.

Since for small σE all superstatistics are the same and given by Tsallis statistics, we also have a maximum entropy principle for the normalized Boltzmann factors of all our superstatistics in terms of the Tsallis entropies [3,18] with an entropic index q given by Eq. (25). For sufficiently small variance of the fluctuations they all extremize the Tsallis entropies subject to given constraints, no matter what the precise form of $f(\beta)$ is.

However, differences arise if σE is not small. Then there are many different superstatistics described by different $B(E)$, and the densities given by $B(E)$ do not extremize the Tsallis entropies in general. Rather, as shown very recently by Tsallis and Souza [25], they extremize more general classes of entropy-like functions. The Boltzmann factor of the examples of superstatistics that we considered above, if expressed in terms of the universal parameters q and β_0, can be written as

$$B(E)=e^{-\beta_0 E}\left(1+\tfrac{1}{2}(q-1)\beta_0^2E^2+g(q)\beta_0^3E^3+\cdots\right)\ , \tag{26}$$

where the function $g(q)$ depends on the superstatistics chosen. We obtain

$$g(q)=0 \text{ (uniform and 2-level)} \tag{27}$$

$$=-\frac{1}{3}(q-1)^2 \text{ (Gamma)} \tag{28}$$

$$=-\frac{1}{6}(q^3-3q+2) \text{ (log-normal)} \tag{29}$$

$$=-\frac{1}{3}\,\frac{(q-1)(5q-6)}{3-q}\ (F \text{ with } v=4)\ . \tag{30}$$

For sufficiently large variance σ^2 or energy E one can distinguish between Tsallis statistics and other superstatistics by the third-order term $g(q)\beta_0^3E^3$, which describes the first significant differences between the various superstatistics and hence provides more information on the underlying complex dynamics. In the turbulence applications described theoretically in Refs. [5,11,12] and experimentally in Refs. [7,10], the fluctuating intensive quantity is related to the energy dissipation rate of the flow, and the fluctuations of this quantity are indeed large. Moreover, the energy E essentially corresponds to the square of local velocity differences in the fluid, and these are also measured up to very large values. Therefore these experiments can well provide precise information about the kind of superstatistics needed to describe the complex dynamics.

An interesting question is what type of averaging process is the physically most relevant one for a generic system with intensive parameter fluctuations. We may either work with unnormalized Boltzmann factors $e^{-\beta E}$ that are averaged over β and then finally normalize by doing the integration over all energy states E (type-A superstatistics), or we may work with locally normalized distributions $p(E) = 1/(Z(\beta))e^{-\beta E}$ and average those over all β (type-B superstatistics). Since in general the normalization constant Z depends on β, the result will differ slightly. However, case B can be easily reduced to case A by replacing the distribution $f(\beta)$ by a new distribution $\tilde{f}(\beta) := CZ(\beta)^{-1}f(\beta)$, where C is a suitable normalization constant. In other words, type-B superstatistics with f is equivalent to type-A superstatistics with $\tilde{f}$. Our formula (25) relating q and the variance of the β fluctuations is valid for both type-A and type-B superstatistics, just that all expectations $\langle \cdots \rangle$ must be formed with either f (type-A) or $\tilde{f}$ (type-B). For example, if $E = \frac{1}{2}v^2$, then $Z(\beta) = \sqrt{2\pi/\beta}$ and hence $\tilde{f}(\beta) \sim \beta^{1/2} f(\beta)$. If $f(\beta)$ is given by the Γ-distribution Eq. (11), then type-A superstatistics yields $q = 1 + 1/c$, whereas type-B yields $q = 1 + 2/(2c + 1)$.

In general, one may look for certain distinguishing properties that select out of the many possible superstatistics the physically most relevant one. For example, Tsallis and Souza [25] show that type-A superstatistics based on Γ-distributions (leading to ordinary Tsallis statistics) is distinguished by the fact that the partition function of the canonical ensemble does not depend on the Lagrange parameter related to normalization, when the corresponding entropy function is maximized. In general, we think that the answer to the question which superstatistics is the most suitable one will often depend on the physical problem under consideration. Different nonequilibrium systems may require different types of superstatistics, depending on the underlying microscopic dynamics.

To summarize, in this paper we have considered a class of generalized statistics, which we have called 'superstatistics'. Tsallis statistics is a special case of these superstatistics. The dynamical parameter q of Eq. (25) can be defined for all these new statistics. For small variance of the fluctuations we have shown that there is universal behavior of all superstatistics. For large variance there are differences which provide information on the underlying complex dynamics. In general, complex nonequilibrium problems may require different types of superstatistics. Tsallis statistics is just one example of many possible new statistics. There is no a priori reason to expect that other superstatistics would not be present in nature, thus confirming the relevance of Einstein's remark.

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