Gödel-Type Universes and Chronology Protection in Hořava-Lifshitz Gravity

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HL GRAVITY AND CAUSAL ANOMALIES

We are investigating the following questions:

- HL quantum gravity permits?
- Gödel-type solutions?
- Gödel-type space-time regions with closed time-like curves?
- Gödel-type space-times with physically well motivated matter content?

Or HL quantum gravity somehow

- incorporates a chronology protection for Gödel-type space-times?
HOŘAVA-LIFSHITZ GRAVITY

• **Quantum Formulation:**
  
  Power-counting renormalizable;
  Ultra-high energy scales (trans-Planckian).

• **Causal Structure of Space-Time:**
  
  ADM formulation (space and time splitting);
  Lifshitz anisotropic scaling (no local Lorentz invariance).

• **Compatibility with General Relativity (GR)**
  
  GR to be recovered at low and medium energy scales (sub-Planckian).

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VIOLATION OF CAUSALITY IN GENERAL RELATIVITY

• Gödel-type Space-Times

\[ ds^2 = -[dt + H(r) \, d\phi]^2 + dr^2 + D(r)^2 \, d\phi^2 + dz^2 \]

• ST-Homogeneity

\[ \frac{1}{D} \frac{dH}{dr} = 2\omega \quad \text{(m, } \omega \text{) are constants such that } \omega^2 > 0 \text{ and } -\infty \leq m^2 \leq \infty \]

\[ \frac{1}{D} \frac{d^2 D}{dr^2} = m^2 \]

Identical pairs \((m^2, \omega^2)\) determine isometric Gödel-type space-times.

\[ m^2 = 2\omega^2 \Rightarrow \text{Gödel geometry!!} \]

• Closed Time-like curves

\[ ds^2 = -dt^2 - 2H(r) \, dt \, d\phi + dr^2 + G(r) \, d\phi^2 + dz^2 \]

Gödel circles \(t, z, r = \text{const}\), when \(ds^2 = G(r) \, d\phi^2 < 0\)

\[ G(r) = D^2(r) - H^2(r) < 0 \]

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VIOLATION OF CAUSALITY IN GENERAL RELATIVITY

- Classes of Godel-type Space-Times
  \[ ds^2 = -[dt + H(r) \, d\phi]^2 + dr^2 + D(r)^2 \, d\phi^2 + dz^2 \]

- Hyperbolic Class
  \[ m^2 > 0 \quad H(r) = \frac{4\omega}{m^2} \sinh^2 \left( \frac{m \, r}{2} \right), \quad D(r) = \frac{1}{m} \sinh(m \, r) \]

- Critical radius
  \[ t, z, r = \text{const,} \quad G(r) < 0 \quad \text{for} \quad r > r_c \]
  \[ \sinh^2 \left( \frac{mr_c}{2} \right) = \left[ \frac{4\omega^2}{m^2} - 1 \right]^{-1} \quad \text{for} \quad 0 < m^2 \leq 4\omega^2 \]

- Causal circle
  \[ G(r) > 0 \]

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VIOLATION OF CAUSALITY IN GENERAL RELATIVITY

- **Trigonometric Class**

\[ m^2 = -\mu^2 < 0 \quad H(r) = \frac{4\omega}{\mu^2} \sin^2 \left( \frac{\mu r}{2} \right), \quad D(r) = \frac{1}{\mu} \sin(\mu r) \]

- **Critical radius**

\[
\begin{align*}
    r_1^{(n)} &= \frac{2\pi n}{\mu}, \quad n = 0, 1, 2, \ldots \\
    r_2^{(n)} &= -\frac{2}{\mu} \left[ \arcsin \left( \frac{\mu}{\sqrt{4\omega^2 + \mu^2}} \right) - \pi n \right], \quad n = 1, 2, \ldots \\
    r_3^{(n)} &= \frac{2}{\mu} \left[ \arcsin \left( \frac{\mu}{\sqrt{4\omega^2 + \mu^2}} \right) + \pi n \right], \quad n = 0, 1, 2, \ldots
\end{align*}
\]

- **Causal circles (alternating with non-causal circles)**

\[ R_1 = \left\{ r \mid (r_1^{(0)} = 0) \leq r \leq r_3^{(0)} \right\}, \quad R_n = \left\{ r \mid r_2^{(n-1)} \leq r \leq r_3^{(n-1)} \right\}, \quad n = 2, 3, \ldots \]

\[ G(r) \times r \]

\[ G(r) > 0 \]

V Workshop Challenges of New Physics in Space
VIOLATION OF CAUSALITY IN GENERAL RELATIVITY

- **Linear Class**

  \[ m^2 = 0 \quad H(r) = \omega r^2, \quad D(r) = r \]

- **Critical radius**

  \[ r_c = \frac{1}{\omega} \]

- **Causal circle**

  \[ G(r) > 0 \]

  \[ r < r_c \]
CAUSAL STRUCTURE OF SPACE-TIME

Two interconnected physically significant ingredients:

1 - the **space-time geometry**, which may include non-causal regions;

2 - the **gravity theory**, which involves the **field equations** and the **matter source**.

We investigate Gödel-type models in HL gravity in two steps:

1 - Gödel-type geometries in the **ADM framework** of HL gravity;

2 - Solutions of HL gravity with Gödel-type spacetime metrics and their **matter contents**.
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DYNAMICAL VARIABLES OF HL GRAVITY

Framework of HL gravity:
ADM splitting of space-time in space and time

Space-time metric recast in the ADM framework:

\[ ds^2 = \ g_{ij} \ dx^i \ dx^j = -N^2 \ dt^2 + g_{ij} (dx^i + N^i \ dt)(dx^j + N^j \ dt) \]

\((i, j = 1, 2, 3)\)

ADM dynamical variables:

The lapse function: \( N(t, \vec{x}) \);
The shift vector: \( N^i(t, \vec{x}) \);
The spatial metric: \( g_{ij}(t, \vec{x}) \).
GÖDEL-TYPE METRICS IN HL GRAVITY

ADM restrictions on the Gödel-type metric:

- The lapse is a real function: $N = \frac{D(r)}{\sqrt{G(r)}} \in \mathbb{R}$;
- The spatial metric is positive defined: $\det(g_{ij}) = \sqrt{G(r)} > 0$;
- The metric function $G(r)$ is positive defined: $G(r) > 0$.

ADM restrictions on closed time-like curves:

- Gödel circles $t, z, r = \text{const}$ where $ds^2 = G(r) d\phi^2 < 0$ are Not allowed.
- Non-causal regions: Not allowed.
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CHRONOLOGY PROTECTION IN HL GRAVITY

- Quantum effects prevents violations of causality (Hawking, 1992).
- A quantum gravity theory:
  - Incorporates a chronology protection;
  - Excludes closed time-like curves.

The ADM framework of HL quantum gravity permits:
- Gödel-type metrics only on the chronology respecting space-time regions.

The ADM framework of HL quantum gravity excludes:
- The causal anomalies of all Gödel-type space-time metrics.

This result is valid for any theory in the ADM framework.
GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

Now we arrive at our second step:

Solutions of HL gravity with Gödel-type spacetime metrics and their matter contents.

$\mathcal{M}^2 > 0$. HL gravity admit Gödel-type solutions in the chronology preserving regions?

We examine this question for the hyperbolic class ($m^2 > 0$), which has the most important solutions in GR:

1. The Gödel solution, where $m^2 = 2\omega^2$.

2. The only causal Gödel-type solution, where $m^2 = 4\omega^2$.

The matter content considered is a perfect fluid.
GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

To simplify the calculations we define new (Cartesian)\( t', x, y, z' \) coordinates:

\[
\tan\left[\frac{\phi}{2} + \left(\frac{m^2}{4\omega}\right)(t' - t)\right] = e^{-mr} \tan(\phi/2),
\]

\[
e^{mx} = \cosh(mr) + \sinh(mr) \cos \phi,
\]

\[
m_y e^{mx} = \sinh(mr) \sin \phi,
\]

\[
z' = z,
\]

where the Gödel-type metric is given by

\[
ds^2 = -\left[dt' + \left(\frac{2\omega}{m}\right) e^{mx} \, dy\right]^2 + e^{2mx} \, dy^2 + dx^2 + dz'^2
\]
The ADM Variables for the Gödel-type metric in Cartesian coordinates

\[ N = \frac{1}{v}, \quad v = \sqrt{1 - \left(\frac{2\omega}{m}\right)^2} \]

\[ N_i = (0, -(2\omega/m) e^{mx}, 0) \]

\[ g_{ij} = \text{diag}(1, G(x), 1) \quad G(x) = v^2 e^{2mx} \]

The chronology preserving interval

\[ m^2 > 4\omega^2 \]
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GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

The Lagrangian for the HL gravity we consider is

\[ L = \sqrt{g} N \left[ \frac{2}{\kappa^2} \left( K_{ij} \dot{K}^{ij} - \lambda K^2 \right) - \frac{\kappa^2}{2w^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2w^2} \sqrt{g} R_{il} \nabla_j R^l_k - \frac{\kappa^2 \mu^2}{8} R_{ij} \dot{R}^{ij} + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( \frac{1 - 4\lambda}{4} R^2 + \Lambda R - 3\Lambda^2 \right) + \mathcal{L}_m \right], \]

where

\[ K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \quad \text{and} \quad C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}} \nabla_k (R^j_l - \frac{1}{4} R \delta^j_l). \]

\(\Lambda\) is a cosmological constant, \(\kappa^2\) is a gravitational constant, \(\lambda, w, \mu\) are coupling parameters of the theory, and \(\mathcal{L}_m\) is the matter lagrangian.
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GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

The HL Field Equations

\[ -4 m^4 \tau v \zeta + 2 \Lambda m^2 \tau v \zeta + 3 \Lambda^2 \tau v \zeta \]
\[ + 2 m^4 v \zeta + 2 \rho v^2 - 4 \omega^2 v - 2 \rho - 2 p = 0 \]
\[ 2 \omega (p v - 4 m^2) = 0 \]
\[ -4 m^4 \tau \zeta - 3 \Lambda^2 \tau \zeta + 2 m^4 \zeta - p v + 4 \omega^2 = 0 \]
\[ -4 m^4 \tau v \zeta + 2 m^4 \zeta - 2 m^4 v \zeta \]
\[ + \rho v^2 - 12 \omega^2 v - \rho - p = 0 \]
\[ 4 m^4 \tau v \zeta - 2 \Lambda m^2 \tau v \zeta - 3 \Lambda^2 \tau v \zeta \]
\[ - 2 m^4 \zeta - p v - 4 \omega^2 = 0 \]

independent parameters \( \rho, \Lambda, \tau, \zeta, \omega, \) and \( m^2 \)

Perfect Fluid with energy-momentum tensor

\[ T^{\mu \nu} = (\rho + p) u^\mu u^\nu + pg^{\mu \nu} \]
GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

Solution of the HL Field Equations

\[ p = \frac{4m^2}{v}, \quad \rho = \frac{8(2\omega^2 - m^2)}{v} \]

\[ \tau = \frac{2m^6}{(\Lambda m^2 + 3\Lambda^2)\omega^2 + 4m^6 - \Lambda m^4}, \]

\[ \zeta = -\frac{1}{\Lambda} \frac{(2\Lambda m^2 + 6\Lambda^2)\omega^2 + 8m^6 - 2\Lambda m^4}{\Lambda m^6 + 3\Lambda^2 m^4} \]

\[ m^2 = \frac{2}{3} \omega^2 \quad \text{and} \quad m^2 = \frac{1}{4} \omega^2 \]

The solution is outside the chronology preserving interval

\[ m^2 > 4\omega^2 \]
GÖDEL-TYPE SOLUTIONS IN HL GRAVITY

Conclusions

HL gravity excludes perfect fluid solution with Gödel-type hyperbolic metrics in the allowed chronology preserving region.

This result holds regardless of the equation of state $\rho/\rho$. 