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Algebraic Structures and the Search for the Theory Of Everything: Clifford algebras, spinors and supersymmetry.

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abstract

These lectures notes are intended to cover a small part of the material discussed in the course "Estruturas algebricas na busca da Teoria do Todo". The Clifford Algebras, necessary to introduce the Dirac's equation for free spinors in any arbitrary signature space-time, are fully classified and explicitly constructed with the help of simple, but powerful, algorithms which are here presented. The notion of supersymmetry is introduced and discussed in the context of Clifford algebras.

1 Introduction

The basic motivations of the course "Estruturas algebricas na busca da Teoria do Todo"consisted in familiarizing graduate students with some of the algebraic structures which are currently investigated by theoretical physicists in the attempt of finding a consistent and unified quantum theory of the four known interactions.

Both from aesthetic and practical considerations, the classification of mathematical and algebraic structures is a preliminary and necessary requirement. Indeed, a very ambitious, but conceivable hope for a unified theory, is that no free parameter (or, less ambitiously, just few) has to be fixed, as an external input, due to phenomenological requirement. Rather, all possible parameters should be predicted by the stringent consistency requirements put on such a theory. An example of this can be immediately given. It concerns the dimensionality of the space-time. We observe four (three spatial and one time) dimensions. However, at present, we are unable to get a consistent quantum theory of gravity if we restrict ourselves to four dimensions. On the other hand, there exists strong theoretical (alas, not experimental) evidence that a consistent theory requires supersymmetry (a beautiful symmetry, discovered thirty years ago, which allows to treat in a unified framework bosonic and fermionic particles) and a total number of ten or, more likely, eleven dimensions (such a theory, still mysterious, is referred to as "M-theory"). The extra, invisible to us, dimensions cannot be observed by present accelerators. The most natural explanation is because they are "too small", i.e. compactified into a small compact manifold (Kaluza and Klein first made such an observation, roughly eighty years ago, when observing that a gravity theory in 5 dimensions reduces to usual gravity coupled to Maxwell's theory, if the extra dimension is compactified to an S^1 circle).

From this example alone it is clear that we need to investigate in full generality arbitrary spacetimes (regardless of their dimensionality and of their signature, i.e. the number of time-like versus space-like dimensions) as well as the structures, like the spinorial fields, living in such space-times.

It is not just space-times which should let be arbitrary. Lie algebras and Lie groups are also fully classified (they are usually presented in terms of their Dynkin diagrams). Generic compact Lie groups are investigated as unification groups containing as a subgroup the $SU(3)\times SU(2)\times U(1)$ group of the standard model. One of these groups, E_8, is naturally associated to the heterotic string, and is a prominent candidate for grand-unification phenomenology.

A first lesson to be learned is that, whenever possible, it is better to formulate our theories in the most general mathematical framework. Then, on physical basis, we restrict the class of theories to be considered according to the various consistency requirements, like respecting Einstein causality, unitarity (it implies preserving probabilities in a quantum theory), etc. Symmetries play a crucial role in this game. In many case their absence can spoil the consistency of a theory. Quantum fluctuations can destroy the original symmetry, originating an anomaly at the quantum level which could be lethal to the consistency of the theory. Quite remarkably, in the standard model of electroweak interactions, the charges of the particles are combined in such a way, for each one of the three given families of particles, to leave the model free of an anomaly (the chiral anomaly associated to the $U(1)$ chiral symmetry), whose presence would spoil the consistency of the theory. The famous $d = 10$ dimension of space-time required by a consistent superstring theory has a similar origin. Only in this particular dimension a symmetry of the quantum superstring, the so-called Weyl symmetry, is not anomalous.

From what said it is clear that, in the search of a unifying "Theory of Everything", we should get acquainted with mathematical methods of classification of algebraic structures. They are crucial for this purpose.

Since it would be impossible to give a meaningful course on strings and M-theory in just 5 lectures, a less ambitious program is mandatory. After briefly recalling the main motivations, we can just focus ourselves and concentrate on analyzing a certain number of structures which are required in the above-mentioned theories. The readers who are interested in knowing more about string theory can consult, e.g., the three books given in the bibliography [1), 2), 3)].

In order to illustrate in a very pedagogical (and also quite visually appealing) example the power of mathematical classification schemes in physics and in geometry, in the first lecture of this course the Euler formula has been introduced and later applied to derive the classification of the platonic solids and other polyhedra. Since this is the content of a separate lecture notes of the author [4)], available at the CBPF library, it will not be repeated here. Similarly, a self-contained lecture notes of the author, concerning division algebras, for those intended to further investigate their connection with Clifford algebras, is also available at the CBPF library [5)].

In the present lecture notes, following the program of the course, the mathematical structures investigated are Clifford algebras and spinorial fields. The reason for this choice will be discussed in more detail in the next section.

Here, it should be mentioned that an attractive feature is that Clifford algebras have been fully classified by mathematicians (quite a time ago) and that physicists can rely on these mathematical results to understand and classify further structures, like generalized supersymmetries, which are at the core of M-theory (M-algebra). In the present text, not only the mathematical classification of Clifford algebras is reviewed. Furthermore, Clifford algebras are explicitly constructed with the help of simple but powerful algorithms. The results here furnished are operational tools for those who are interested in concretely working on strings and M-theory. It should be stressed that some of these results are here presented for the first time.

In a following section the notion of supersymmetry is introduced. Some considerations about supersymmetry, its main properties, as well as its role in the present understanding of the unification program are made. The strict connection between supersymmetry and Clifford algebras will be emphasized.

2 Clifford Algebras and the Dirac Equation

It is useful to recall how Dirac was led to discover his famous equation. The second-order Klein-Gordon equation

$$\partial_\mu \partial^\mu \Phi + m^2 \Phi = 0 \tag{1}$$

which can be considered as the free relativistic analogue of the Schrödinger equation, failed to provide a positive-defined probability, when interpreting the field Φ as a quantum amplitude. Dirac was led to search for a first-order relativistic equation(the Schrödinger equation is of first order in time), which in some sort can be considered the "square root"of the Klein-Gordon equation.

The Dirac equation

$$i\Gamma^\mu \partial_\mu \Psi - m\Psi = 0 \tag{2}$$

is indeed a "square root"of the Klein-Gordon equation if it is possible to find a set of matrices Γ^μ whose anticommutators satisfy the condition

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\eta^{\mu\nu} \tag{3}$$

where $\eta^{\mu\nu}$ denotes the flat $(+---)$ metric of the Minkowski spacetime. The field Ψ is a column vector.

The equation (3) defines a Clifford algebra (for a generic signature p, q, while in the case above we have $p = 1$, $q = 3$).

In contrast with the Klein-Gordon case, the Dirac equation admits a consistent probability interpretation, when considering Ψ as a quantum amplitude. The existence of a tower of positive and negative energy eigenstates (a result of the "square root nature"of the Dirac equation) forced Dirac to interpret the positron as a "hole"in a sea of negative-energy electrons, a consistent vacuum being given by filling with electrons all negative-energy states. By the way, the consistency of this procedure requires the field Ψ describing a Fermionic particle, that is a particle satisfying the Pauli exclusion principle.

Free equations such as the Klein-Gordon and the Dirac equations cannot describe interactions. On the other hand in a relativistic theory interactions are introduced through non-linear equations. However, non-linear equations prevent interpreting the fields as quantum amplitudes (the superposition principle for amplitudes fails to be satisfied in a non-linear theory). In order to make a consistent quantum theory out of a non-linear equation a new prescription has to be given. This is indeed possible in the so-called "second quantization scheme". In this interpretation, the fields entering the Klein-Gordon and the Dirac equation are no longer assumed to be quantum amplitudes. They are considered instead operators acting on given Hilbert spaces of quantum states. In this re-interpretation of the basic equations, the Klein-Gordon equation finds a consistent quantum-mechanical interpretation. The relativistic causality, i.e. the propagation of the information at speed not exceeding the speed of light, requires the Klein-Gordon equation describing bosonic particles while the Dirac equation continues to be applicable to fermionic particles alone (all this is in consequence of the famous spin-statistic theorem).

We have seen so far that such fundamental particles like the electrons that, when coupled to the electromagnetic field, give rise to the astonishing successful theory of quantum electrodynamics, are described by the Dirac equations, and therefore require the introduction of Clifford algebras. Dirac solved the problem of finding a set of four matrices satisfying (3) (otherwise, he would not have produced the Dirac's equation!). A series of natural questions to be asked (for which, thanks to the mathematicians, we fully know the answer) is whether matrix representations of Clifford algebras always exist, which is their size, and if they are unique (up to equivalence). In the next sections the answers to such and related questions will be furnished.

Let me emphasize here that, according to what already discussed in the introduction, to have a full control of Clifford algebras is of paramount importance in order to systematically investigate, in any given signature space-time, the properties of the corresponding Dirac equations. More generally, Clifford algebras enter whenever we have spinorial particles.

3 Clifford algebras: classification and explicit construction.

In the previous sections we have discussed the physical motivation leading us to introduce and investigate Clifford algebras in arbitrary signature spacetimes. This section is very different, a little bit more technical, it presents the mathematical results concerning Clifford algebras. The nice feature is that these results can be presented in a self-contained way. A good reference for physicists on Clifford algebra is given by [6]. The formulas below however are more explicit than those furnished in [6] and can be used to concretely work and investigate field thories in any given signature spacetime.

Two remarks are in order. The first one: despite the fact that quantum theory is described by complex numbers, without loss of generality (complex numbers can be considered as points in the real plane) it is preferable to work with Clifford algebras introduced by real-valued matrices. The structure of Clifford algebras is much clearer in such a framework (e.g. its connection with division algebras properties). A second comment: the algorithm furnished below consents in individuating a single representative for each irreducible class of representations of Clifford's Gamma matrices.

The construction is as follows. Let us prove at first that a recursive construction of $D+2$ spacetime dimensional Clifford algebras is at hand, when assumed known a D dimensional representation. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D = p+q$ spacetime with (p,q) signature (providing a representation for the $C(p,q)$ Clifford algebra) then $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

$$\begin{aligned} \Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) &\mapsto (p+1,q+1). \end{aligned} \tag{4}$$

or

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$$

$$(p,q) \mapsto (q+2,p). \tag{5}$$

Some remarks are in order. The two-dimensional real-valued Pauli matrices τ_A, τ_1, τ_2 which realize the Clifford algebra $C(2,1)$ are obtained by applying either (4) or (5) to the number 1, i.e. the one-dimensional realization of $C(1,0)$. We have indeed

$$\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

All Clifford algebras are obtained by recursively applying the algorithms (4) and (5) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0,3+4m)$ (m non-negative integer), which must be previously known. This is in accordance with the scheme illustrated in the table below.

Table with the maximal Clifford algebras (up to $d=256$).

1	*	2	*	4	*	8	*	16	*	32	*	64	*	128	*	256	*
(1, 0)	⇒	(2, 1)	⇒	(3,2)	⇒	(4,3)	⇒	(5,4)	⇒	(6,5)	⇒	(7,6)	⇒	(8,7)	⇒	(9,8)	⇒
						(1,4)	→	(2,5)	→	(3,6)	→	(4,7)	→	(5,8)	→	(6,9)	→
					↗												
				(0,3)													
					↘												
						(5,0)	→	(6,1)	→	(7,2)	→	(8,3)	→	(9,4)	→	(10,5)	→
								(1,8)	→	(2,9)	→	(3,10)	→	(4,11)	→	(5,12)	→
							↗										
						(0,7)											
							↘										
								(9,0)	→	(10,1)	→	(11,2)	→	(12,3)	→	(13,4)	→
														(1,12)	→	(2,13)	→
													↗				
												(0,11)					
													↘				
														(13,0)	→	(14,1)	→
																(1,16)	→
															↗		
														(0,15)			
															↘		
																(17,0)	→

Concerning the previous table, some remarks are in order. The columns are labeled by the matrix size **d** of the maximal Clifford algebras. Their signature is denoted by the (p, q) pairs. Furthermore, the underlined Clifford algebras in the table are called the "primitive maximal Clifford algebras". The remaining maximal Clifford algebras appearing in the table are the "maximal descendent Clifford algebras". They are obtained from the primitive maximal Clifford algebras by iteratively applying the two recursive algorithms (4) and (5). Moreover, any non-maximal Clifford algebra is obtained from a given maximal Clifford algebra by deleting a certain number of Gamma matrices. It should be noticed that Clifford algebras in even-dimensional spacetimes are always non-maximal.

Let us discuss concretely a given example, namely the explicit construction of the $D = p + q$ spacetime dimensional Clifford algebras for $D = 11$ (the dimensionality of M-theory). We obtain

(p, q)	*type*	d
(11,0)	$\subset$ (11,2)	64
(10,1)	**M**	32
(9,2)	$\subset$ (11,2)	64
(8,3)	**M**	64
(7,4)	$\subset$ (7,6)	64
(6,5)	**M**	32
(5,6)	$\subset$ (7,6)	64
(4,7)	**M**	64
(3,8)	$\subset$ (3,10)	64
(2,9)	**M**	32
(1,10)	$\subset$ (3,10)	64
(0,11)	**M**	32

where the maximal Clifford algebras are labeled by M (the remaining non-maximal algebras are recovered from the maximal ones given on the second column, after deleting a certain number of Γ-matrices. The size of the matrix representation is given by the number on the right (d).

So far we have left unexplained how to explicitly construct the primitive maximal Clifford algebras of the series $C(0, 3+8n)$ (also known as quaternionic series, due to its connection with this division algebra), as well as the octonionic series $C(0, 7+8n)$. The answer can be provided with the help of the three Pauli matrices (6). We construct at first the 4×4 matrices realizing the Clifford algebra $C(0, 3)$ and the 8×8 matrices realizing the Clifford algebra $C(0, 7)$.

They are given, respectively, by

$$C(0,3) \equiv \begin{array}{l} \tau_A \otimes \tau_1, \\ \tau_A \otimes \tau_2, \\ \mathbf{1}_2 \otimes \tau_A. \end{array} \quad (7)$$

and

$$C(0,7) \equiv \begin{array}{l} \tau_A \otimes \tau_1 \otimes \mathbf{1}_2, \\ \tau_A \otimes \tau_2 \otimes \mathbf{1}_2, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_1, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_2, \\ \tau_1 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_2 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_A \otimes \tau_A \otimes \tau_A. \end{array} \quad (8)$$

The three matrices of $C(0,3)$ will be denoted as $\overline{\tau}_i, = 1,2,3$. The seven matrices of $C(0,7)$ will be denoted as $\tilde{\tau}_i$, $i = 1,2,\ldots,7$.

In order to construct the remaining Clifford algebras of the series we need at first to apply the (4) algorithm to $C(0,7)$ and construct the 16×16 matrices realizing $C(1,8)$ (the matrix with positive signature is denoted as γ_9, ${\gamma_9}^2 = \mathbf{1}$, while the eight matrices with negative signatures are denoted as γ_j, $j = 1,2\ldots,8$, with ${\gamma_j}^2 = -\mathbf{1}$). We are now in the position to explicitly construct the whole series of primitive maximal Clifford algebras $C(0,3+8n)$, $C(0,7+8n)$ through the formulas

$$C(0,3+8n) \equiv \begin{array}{lll} \overline{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots & \ldots \otimes \gamma_9, \\ \mathbf{1}_4 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \ldots & \ldots & \ldots, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \ldots & \ldots & \otimes \gamma_9 \otimes \gamma_j, \end{array} \quad (9)$$

and similarly

$$C(0,7+8n) \equiv \begin{array}{lll} \tilde{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots & \ldots \otimes \gamma_9, \\ \mathbf{1}_8 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \ldots & \ldots & \ldots, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \ldots & \ldots & \otimes \gamma_9 \otimes \gamma_j, \end{array} \quad (10)$$

Please notice that the tensor product of the 16-dimensional representation is taken n times. The total size of the (9) matrix representations is then 4×16^n, while the total size of (10) is 8×16^n.

The formulas given above provide the complete set of answers to the questions raised in the previous section. They furnish a concrete and operative method to compute the fully classified Clifford algebras.

It should be noticed that all Clifford matrices are even-dimensional (power of 2). An important subclass of Clifford Gamma matrices is obtained by the matrices which are decomposable in 2×2 blocks and are non-vanishing only in the anti-diagonal blocks. Such matrices can be named as (generalized) Weyl-type matrices. An inspection of the previous tables shows that the set of the (generalized) Weyl matrices is found in special signature dimensions. All primitive Clifford algebras are not of (generalized) Weyl type. However, all the derived Clifford algebras, through the two lifting algorithms, are of Weyl-type, once deleted the $\begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$ matrix to produce a non-maximal Clifford algebra.

To give a concrete example, the two-dimensional Euclidean space $(2,0)$ is not of Weyl type, while the two-dimensional Minkowski spacetime $(1,1)$ is of Weyl type. Indeed, the first one is obtained from the $(2,1)$ Clifford algebra by deleting a space-type Gamma matrix, while the second one is obtained from the same $(2,1)$ Clifford algebra by deleting one of the two temporal-type Gamma matrices. Without loss of generality this Gamma matrix can always be chosen to be given by $\begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$.

That is good, but what is so special about Weyl realizations of Clifford algebras? The reason is the following. The commutator between Gamma matrices, $\Sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu]$, is the generator of the Lorentz group which corresponds to the given signature space-time. The product of two Gamma matrices, both admitting non-vanishing blocks only in the antidiagonal, correponds to a 2×2 block matrix whose only non-vanishing components are in the diagonal blocks. Since both the Gamma matrices, as well as the Lorentz generators $\Sigma_{\mu\nu}$ act on spinors, the fact that the Lorentz generators are block-diagonals, means that we can consistently set, under these conditions, equal to 0 half of the components of the column vector spinors (either the upper half or the lower half), to produce a so-called Weyl spinor, admitting half of the degrees of freedom expected by the original Dirac spinor. This reduction of the components can be operated acting on a Dirac spinor with a projector $P_\pm$ ($P_\pm P_\mp = 0$ and ${P_\pm}^2 = P_\pm$), given by

$$P_\pm = \frac{1}{2}(\mathbf{1} \pm \overline{\Gamma}) \tag{11}$$

where $\overline{\Gamma} = \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}$.

In odd-dimensional space-times the matrix $\overline{\Gamma}$ is always given by the product of all the other Γ matrices. In standard 4-dimensional Minkowski space-time such a product is usually denoted as Γ_5. An important feature is that under parity transformation the space coordinates x_1, x_2, x_3 change their sign ($x_i \mapsto -x_1$ for $i = 1, 2, 3$), as well as all covariant quantities depending on space indices. As already recalled, Γ_5 is obtained by taking the product of the four Gamma matrices of the Minkowski space-time. Three of them are of space type and therefore affected by a parity transformation. As a consequence, under a parity transformation, Γ_5 changes its sign ($\Gamma_5 \mapsto -\Gamma_5$). Γ_5 is a pseudoscalar matrix (for what concerns the standard Minkowski space-time). The Weyl spinors, constructed after a projection involving Γ_5, are not invariant under parity: they are chiral (respectively antichiral) spinors.

In order to construct lagrangian terms, which are scalar under Lorentz transformations and are given by bilinear products of spinors, we need to introduce the notion of barred spinors $\overline{\Psi}$, given by $\Psi^T \cdot A$, where T denotes transposition (remember that in our conventions spinors are withou loss of generality assumed to be real) and A is a matrix, given by the product of all temporal Gamma matrices.

Kinetic (massive) terms are of the form $\overline{\Psi}\Gamma^\mu\partial_\mu\Psi$ and respectively $\overline{\Psi}\Psi$. To be present in a lagrangian, these quantities of course must be non-vanishing. However, spinors are described by Grassmann (anticommuting fields) to take into account of the fact, already discussed, that they obey the Pauli exclusion principle. For a generic Grassmanm field ψ is such that $\psi^2 = 0$.

A little inspection shows that, when dealing with Weyl spinors, we cannot be assured of the non-vanishing of the above bilinear terms. A very instructive exercise which is left to the reader consists in computing which are the spacetimes which allow the existence of massive Weyl spinors.

4 Supersymmetry

We have recalled that in nature two kinds of particles exist. The bosonic and the fermionic ones. One of the great successes of Quantum Field Theory is that it correctly predicts the nature of any given particle associating its statistics (bosonic or fermionic) to its spinorial properties.

From the point of view of a unified theory, it would be quite natural searching for an explanation for both types of particles properties in a single, unifying framework. Different possibilities are opened. We can think to use the property that two fermions made up one boson (a consequence of the tensor properties of spinors, such as $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$), in order to try interpreting bosons as bounded states of fermions. This possibility, which received a lot of attention, is nowadays not too much attracting.

Another, much more appealing possibility, consists in investigating a symmetry which allows bosons and fermions to be inserted in the same multiplet. Such a symmetry is known as "supersymmetry".

In the sixties, it was believed that it is not possible to have symmetries with particles of different spin in the same multiplet. A famous no-go theorem, the Coleman-Mandula theorem, prevented such a thing to happen. However, no-go theorems are valid as long as their hypothesis are satisfied. Relaxing the involved hypothesis can pave the way to overrule a no-go theorem. Indeed, the Coleman-Mandula theorem did not contemplate the extension of symmetry provided by graded algebras, leading to supergroups, constructed with both commuting and anticommuting (Grassmann) variables. Such an extension allows particles of integer and half-integer spin (namely bosons and fermions, which are described by usual commuting and, respectively, anticommuting fields) to be accommodated into a single multiplet of supersymmetry transformations.

Physicists discovered in the seventies the theoretical feasibility of supersymmetry (so far, no experimental evidence of supersymmetry has been observed). The Haag-Lopuszanski-Sohnius theorem, providing the classification of supersymmetry algebras, replaced the Coleman-Mandula no-go theorem. The history is a bit twisted, because the classification of supersymmetry algebras provided by Haag-Lopuszanski-Sohnius in the seventies is not complete either. It does not take into account, e.g., the existence of extended objects, which lead, in $D = 11$ dimensions, to the M-algebra.

Anyway, in the seventies the supersymmetry was discovered and revealed its nice features. The presence of "miraculous"(due to supersymmetry and the opposite contribution of bosonic and fermionic particles) cancellations of divergencies in Feynman graphs gave hope to the possibility of finding a consistent quantum theory of gravity. Indeed supergravity is "better behaved"with respect of ordinary gravity, in this respect, but it is still not the final answer (supergravity,like gravity, is a non-renormalizable theory and not all divergent Feynman graphs are cancelled by supersymmetry).

However, supersymmetry is required in the context of string theory. It allows to confine in an unphysical sector the tachyonic state present in string theories. The ordinary bosonic string theory implies the existence of a tachyon, whose presence, however, is problematic due to the violation of the Einstein causality.

The closed 10 dimensional superstring theory provides a theory which contains gravity and it is finite at all order of its perturbation expansion, given by sum over different genus Riemann surfaces.

In order to introduce what supersymmetry is, it is better to define it at first in its simplest framework, that of the 1-dimensional supersymmetric quantum mechanics. In this context the supersymmetry is characterized by the existence of an operator Q, which is the square root of the hamiltonian H, namely

$$Q^2 = H \tag{12}$$

Let us take such an operator in the form of a hermitian matrix given by

$$Q = \begin{pmatrix} 0 & P \\ P^\dagger & 0 \end{pmatrix} \tag{13}$$

Therefore the hamiltonian H is given by

$$Q = \begin{pmatrix} PP^\dagger & 0 \\ 0 & P^\dagger P \end{pmatrix} \tag{14}$$

A fermion number operator F can be defined according to

$$F = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{15}$$

The following (anti)-commutation relations are satisfied

$$[H, Q] = [H, F] = 0 \tag{16}$$

and

$$\{Q, F\} = 0 \tag{17}$$

(the latter equation involves the anticommutator, being both Q and F fermionic operators).

The "bosonic states" can be defined as those admitting $+1$ eigenvalue of the fermionic number operator F, while the fermionic states are eigenvectors corresponding to the -1 eigenvalue.

For such a given supersymmetric hamiltonian H, it is easily proven that any positive eigenvalue of the energy admits a degenerate pair of eigenvectors, which are in one-to-one correspondence, one living in the bosonic sector, the other in the fermionic sector. This is a plain consequence of the associativity property

$$P(P^\dagger P) = (PP^\dagger)P \tag{18}$$

Indeed, if Φ_n is an eigenvector of $P^\dagger P$ with eigenvalue $\lambda_n > 0$, then $\Psi_n = P\Phi_n$ is an eigenvector of $PP^\dagger$ for the same eigenvalue λ_n (and conversely).

This one-to-one correspondence between "bosonic" and "fermionic" states is possibly broken for 0-energy (in this case in fact $P\Phi_n = 0$). A non-trivial index $I \neq 0$ can appear whenever the number of bosonic eigenvectors describing a degenerate vacuum state is different from the number of fermionic eigenvectors

$$I = dim(KerP) - dim(KerP^\dagger) \tag{19}$$

Index theorem arising from such constructions are very important, both in pure mathematics and in physics (where they are associated to the anomalies, already discussed in the introduction).

A lesson that we have learned from this very simple, one-dimensional, quantum mechanical example, is that in a supersymmetric theory bosonic and fermionic states are paired (apart possible anomalies). Moreover, in one dimension, an equal number of bosonic and fermionic states is found for any given eigenvalue $\lambda > 0$ of the energy.

In a higher dimensional relativistic theory such a result finds a counterpart. In this case it is no longer the energy, but the mass of the particles (a Lorentz-invariant quantity) that enters the equation. In a supersymmetric theory an equal number of bosonic and fermionic states are found at each level of the mass. Stated otherwise, bosonic and fermionic particles are coupled. To any given ordinary particle is associated its supersymmetric partner, of opposite statistics and equal mass (due to the spin-statistics theorem superpartners differ by half-integer spin). This of course raises a natural question. Where have to be found the supersymmetric partners of known particles, since they have not been observed? The commonly accepted explanation is that supersymmetry is a spontaneously broken symmetry, namely it admits a set of degenerate vacuum states, which are not left invariant by supersymmetry. Under this condition the supersymmetric partners of the ordinary particles become more massive than their ordinary partners (the mass-difference being related to the scale of breaking of supersymmetry). A spontaneously broken supersymmetry can account for the fact that supersymmetric partners of the ordinary matter have not been experimentally observed yet.

A relativistic supertranslation algebra which replies in higher dimensions the (12) relation requires being formulated in a relativistic covariant way. The energy H is in this case the time-component of the P_μ momentum. The vector index μ must be saturated to produce a scalar quantity. An obvious way of saturating it consists in multipliying P_μ with the Clifford Gamma matrix Γ^μ (and taking the summation over repeated indices according to the Einstein convention).

For what concerns the supersymmetry generators, they are spinorial quantities and they are expected to carry spinorial indices. Therefore the relativistic supertranslation algebra in any given spacetime, can be postulated to be given by

$$\{Q_a, Q_b\} = (P_\mu \Gamma^\mu)_{ab} \tag{20}$$

The spinorial indices a, b take value $a, b = 1, \ldots, d$, where d is the size of the corresponding Clifford algebra (easily computed with the help of the previous section results). It should be noticed that, while the vector indices μ, ν grow linearly when increasing the dimensionality D of the spacetime, $\mu, \nu = 1, \ldots, D$, the spinorial indices grow as a power in D, we have indeed roughly $d \sim 2^{\frac{D}{2}}$ (different extra factors are assigned to specific signature spacetimes, according to the previous section results, an extra $\frac{1}{2}$ factor is present for Weyl spinors, etc.). If we remember the conclusion that supersymmetry requires an equal number of bosons and fermions, it is clear that the supersymmetrization of a given theory is not always possible, unless in some specific dimensions. In general there exists a maximal space-time dimensionality allowing the construction of a supersymmetric theory. The maximal superYang-Mills theory exists in $D = 10$ dimensions, while the maximal supergravity is allowed for $D = 11$. Supergravity theories can be constructed in $D \leq 11$ spacetimes, however no supergravity exists in $D > 11$. This is of course a strong indication that something special happens in $D = 11$, where the M-theory (admitting $D = 11$ supergravity as low-energy limit) is searched.

When we dimensionally reduce the extra-dimensions down to, let's say, $D = 4$, we obtain an N-extended supersymmetry (N counts the number of supersymmetries present in the model). The dimensional reduction of the $D = 10$ superYang-Mills theory leads to the maximally extended $N = 4$ superYang-Mills theory in $D = 4$ dimensions, while the dimensional reduction of the $D = 11$ supergravity leads to the maximally extended $N = 8$ supergravity in $D = 4$.

If we further dimensionally reduce (i.e. we freeze the dependence on them) the three spatial coordinates, we end up with an extended supersymmetric quantum mechanical systems admitting $N = 16$ (for superYang-Mills), and respectively $N = 32$ (for supergravity) supersymmetries.

Supersymmetrically N-Extended Quantum Mechanical systems are generalizations of (12) (which corresponds to $N = 1$) and they are given by the algebra

$$\{Q_i, Q_j\} = \delta_{ij} H \tag{21}$$

where H is the hamiltonian, as before, and $i, j = 1, \ldots, N$. In such theories we have N supersymmetry generators Q_i.

Such an algebra is quite reminiscent of an Euclidean Clifford algebra. Indeed, according to the results of [7)], there exists a one-to-one correspondence between Euclidean Clifford algebra of Weyl type (I remember, not vanishing blocks in the antidiagonal) and supersymmetrically extended quantum mechanical systems. The correpondence is such that

$$D = N \tag{22}$$

and

$$d = n \tag{23}$$

where, on the left hand side, D and d denote respectively the dimensionality of the Euclidean space and the size of the corresponding Clifford Gamma matrices, while in the right hand side N and n denote the number of extended supersymmetries and the number of states (bosonic and fermionic) in a given finite irreducible multiplet of representation of the extended supersymmetry.

The correspondence is made precise if, starting from a Weyl representation of the Clifford algebra

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i & 0 \end{pmatrix} \tag{24}$$

we represent the N supersymmetric generators of the extended quantum mechanical system as

$$Q_i = \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i H & 0 \end{pmatrix} \tag{25}$$

It is immediately checked that the generators above satisfy the N-extended supersymmetry algebra.

The converse is true as well. From the supersymmetry generators we end up with a Weyl realization of the corresponding Clifford algebra.

Let me end this section concerning supersymmetry with a final step, the introduction of the M-algebra in $D = 11$ dimensions. In the $D = 11$ Minkowskian spacetime (with signature $(10, 1)$), from the previous section results, spinors have 32 components.

When we take the anticommutator of two real spinors, as in (20), the most general result that we can expect consists of a 32×32 symmetric matrix, which admits $32 + \frac{32\cdot 31}{2} = 528$ components. On the other hand, the right hand side of (20) is given by only 11 components and by no means saturates the number of 528. Where is it possible to accommodate the extra generators that should be expected in the right hand side? Taking the totally antisymmetrized product of k Gamma matrices (there are $\begin{pmatrix} D \\ k \end{pmatrix}$ such kind of objects) we can construct the most general $d \times d$ matrix. The requirement on the matrix of being symmetric implies that the total number of 528 is obtained by summing the $k = 1$, $k = 2$ and $k = 5$ sectors, so that $528 = 11 + 55 + 462$. The most general supersymmetric algebra in $D = 11$ can therefore be presented as

$$\{Q_a, Q_b\} = (C\Gamma_\mu)_{ab} P^\mu + (C\Gamma_{[\mu\nu]})_{ab} Z^{[\mu\nu]} + (C\Gamma_{[\mu_1 \ldots \mu_5]})_{ab} Z^{[\mu_1 \ldots \mu_5]} \tag{26}$$

(where $C = A$ is the charge matrix).

$Z^{[\mu\nu]}$ and $Z^{[\mu_1 \cdots \mu_5]}$ are tensorial central charges, of rank 2 and 5 respectively. Each central term on the right hand side corresponds to an extended object, a p-brane. The algebra (26) is also called M-algebra. It provides the generalization of the ordinary supersymmetry algebra (20).

5 Conclusion

In this talk we have introduced some of the mathematical structures needed to investigate the unification program of the interactions as presently understood. We have briefly mentioned the motivations why we are looking for extended space-times and why we are forced to investigate our mathematical structures in the most systematic and general form as possible.

The structures here analyzed include the Clifford algebras, the spinors and the supersymmetry. For what concerns Clifford algebras we have here furnished a complete set of results, providing an algorithm to explicitly compute Clifford algebras in any given signature spacetime. In the following, we showed how the properties of Clifford algebras are reflected in the properties for the spinorial fields satisfying a generalized (in any space-time) Dirac equation. The notion of Weyl spinor has also been introduced and some of its basic properties have been analyzed.

In the following, the acquired knowledge on spinors, allowed to introduce the notion of supersymmetry. At first this was done in a one-dimensional quantum mechanical system, and later in the relativistic case, for any given spacetime. The notion of extended supersymmetry has been introduced and the connection between extended supersymmetries and Clifford algebras (of Weyl type) has been pointed out. As a final result the $D = 11$ dimensional algebra of generalized supersymmetries, known as M algebra, admitting the presence of tensorial central charges associated to extended objects, has been written down.

It should be pointed out that an important topic discussed during the lectures, has not been addressed here. It concerns the connection of Clifford algebras and extended supersymmetries with the division algebras of the real, complex, quaternionic and even octonionic numbers. This topic has not been inserted in the present text since it has already been discussed by the author in the monografia (Notas de aula) ([5]). The fundamental reference concerning the connection between division algebras and extended supersymmetries is given by ([8]).

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