1. A particle is an observer-dependent concept (even in flat space-time!). This becomes manifest when investigating the excitation of a particle detector moving in a space-time. A particle detector may be simply modeled as a quantum system with discrete energy levels \(|E_i|\), for \(i \in \mathbb{N}\) (say, an ideal atom). The coupling of the detector to a real scalar field \(\varphi\) is described by the interaction Hamiltonian \(c \hat{n}(\tau)\hat{\varphi}(x(\tau))\), where \(\hat{n}(\tau)\) is the detector’s monopole moment operator, \(c \in \mathbb{R}\) is a coupling constant and \(x(\tau)\) is the trajectory of the detector parameterized by its proper time \(\tau\). Then, one can show that, if the coupling constant \(c\) is small, the transition probability rate from a configuration in which the field is in a state \(|\psi\rangle\) and the detector in a state \(|E_i\rangle\) to any other configuration is given by the following first-order perturbation expression:

\[
\frac{dP_{E_i \rightarrow E_j}}{dt} = \sum_j c^2 |\langle E_i | \hat{n}(0) | E_j \rangle|^2 \Pi_{\Delta E},
\]

where \(\Delta E = E_j - E_i\) and \(\Pi_{\Delta E}\) is the response function given by

\[
\Pi_{\Delta E} = \int_{-\infty}^{\infty} d\tau e^{-i\tau \Delta E} G_+(x(\tau), x(0)),
\]

where \(G^M_+(x, x') = \langle \psi | \hat{\varphi}(x) \hat{\varphi}(x') | \psi \rangle\) is the Wightman function in the state \(|\psi\rangle\). Note that the response function does not depend on the internal structure of the detector, but does depend on its trajectory.

Let us consider a massless, real scalar field \(\varphi\) which is minimally coupled to the scalar curvature (i.e., coupling constant \(\xi = 0\)) on \((3 + 1)\)-D flat space-time and Cartesian coordinates \((t, \vec{x})\).

(a) Calculate the response function \((1)\) of a detector following an inertial trajectory \(\vec{x}(t) = \vec{x}_0 + \vec{v} t\) (where \(\vec{v}\) is a constant 3-velocity and \(|\vec{v}| < 1\) in units where \(c = 1\)) when the field is in the Minkowski vacuum \(|M\rangle\). Remember that the proper time of an inertial observer \((\tau)\) and the Minkowski time \((t)\) are related by \(t = \frac{1}{\sqrt{1 - |\vec{v}|^2}} \tau\) and that the above Wightman function is:

\[
G^M_+(x, x') = \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi^2} \frac{1}{-(\Delta t - i\epsilon)^2 + |\vec{x} - \vec{x}'|^2}. \tag{2}
\]

(b) Calculate the response function \((1)\) for a detector following a uniformly-accelerated trajectory, with acceleration \(a\) along the \(x\)-axis, when the field is in the Minkowski vacuum. Remember that this (Rindler) trajectory is hyperbolic in the \((x, t)\)-plane: \(-t^2 + x^2 = 1/a^2; y = z = 0\), and the proper time of a Rindler observer \((\tau)\) and the Minkowski time \((t)\) are related by \(t = \sinh(\alpha \tau)/\alpha\). You shall find that the response function contains a thermal factor, which means that a particle detector in a Rindler trajectory sees a thermal bath when the field is in the Minkowski vacuum.

2. Consider a real massless scalar field \(\varphi\) in \((1+1)\)-D flat space-time and Cartesian coordinates \(t\) and \(y\). Let us model a mirror in this space-time as a point on which the field vanishes, i.e., if the trajectory of the mirror is \(y = y_m(t)\), then the mirror is represented by the boundary condition

\[
\varphi(t, y_m(t)) = 0. \tag{3}
\]

We require that \(|y_m(t)| < 1\), so that the mirror travels on a time-like path \((c = 1)\), and that \(y_m(t) = 0\) for \(t < 0\). In null coordinates \(u = t - y\) and \(v = t + y\), we write the trajectory of the mirror via some function \(p\) as \(v = p(u)\), and the field equation can be written as

\[
\Box \varphi = \frac{\partial^2 \varphi}{\partial u \partial v} = 0. \tag{4}
\]

On and to the right of the path of the mirror \((y \geq y_m)\), a complete set of orthonormal mode solutions of eq.(4) satisfying the condition \((3)\) is given by

\[
\varphi^{in}_\omega(u, v) \equiv \frac{1}{\sqrt{4\pi \omega}} \left( e^{-i\omega v} - e^{-i\omega p(u)} \right), \tag{5}
\]
where $\omega > 0$. The quantized field $\hat{\varphi}$ can be expanded in terms of this set as

$$\hat{\varphi} = \int_0^\infty d\omega \left( a_{\omega}^{in} \varphi_{\omega}^{in} + a_{\omega}^{in\dagger} \varphi_{\omega}^{in*} \right),$$

and the in-vacuum $|in\rangle$ can be defined via $a_{\omega}^{in}|in\rangle = 0, \forall \omega > 0$.

(a) Using the fact that $y_m(t) = 0$ for $t < 0$, show that the in-modes given by eq.(5) are positive-frequency with respect to Minkowski time $t$ for $t < 0$. This means that the in-vacuum $|in\rangle$ coincides with the Minkowski vacuum $|M\rangle$ for $t < 0$, where $|M\rangle$ is defined, as usual, expanding the field $\hat{\varphi}$ in terms of the set (6) below.

(b) Calculate the Wightman function in the in-vacuum, $G_{\omega}^{|in\rangle}(x,x') = \langle in|\hat{\varphi}(x)\hat{\varphi}(x')|in\rangle$. You will need to regularize an integral, which you may do as indicated in part (b).

(c) In flat space-time without the mirror, a complete set of orthonormal solutions, positive-frequency with respect to $t$, are the usual Minkowski modes

$$\varphi_k^M(x) \equiv \frac{1}{\sqrt{4\pi\omega}} e^{i(ky - \omega t)} = \begin{cases} \varphi_\omega^M(u,v) \equiv \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u}, & \text{if } k > 0 \\ \varphi_\omega^V(u,v) \equiv \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v}, & \text{if } k < 0 \end{cases},$$

where $\omega = |k|, k \in \mathbb{R}$. Calculate the Wightman function in the Minkowski vacuum, $G_{\omega}^{|M\rangle}(x,x') = \langle M|\hat{\varphi}(x)\hat{\varphi}(x')|M\rangle$. You will again need to regularize an integral, which you may do as indicated in part (b).

(d) Now, calculate the quantity

$$\langle in| : \hat{\varphi}(x)\hat{\varphi}(x') : |in\rangle = \langle in|\hat{\varphi}(x)\hat{\varphi}(x')|in\rangle - \langle M|\hat{\varphi}(x)\hat{\varphi}(x')|M\rangle.$$

After that, you can take the limit $\varepsilon \to 0^+$.

(e) From the classical expression of the stress-energy tensor $T_{\mu\nu}$ of a real, massless and minimally-coupled (i.e., coupling constant $\xi = 0$) scalar field, one finds:

$$T_{uu} = \partial_u \varphi \partial_u \varphi, \quad T_{vv} = \partial_v \varphi \partial_v \varphi, \quad T_{uv} = T_{vu} = 0.$$

Given these expressions, calculate the renormalized expectation value $\langle in| : T_{\mu\nu}(x) : |in\rangle$ of the stress-energy tensor in the in-vacuum, using the expression calculated in (7).

Hint: the required renormalized expectation value can be calculated via:

$$\langle in| : T_{uu}(x) : |in\rangle = \lim_{x \to x'} \partial^2_{uu} \langle in| : \hat{\varphi}(x)\hat{\varphi}(x') : |in\rangle,$$

and similarly for the other components.

3. The Casimir effect is an example where the effects of boundary conditions demonstrate the non-uniqueness of the vacuum state. Consider a real, massless, scalar field $\varphi$ which is conformally coupled to the scalar curvature (coupling constant $\xi = 1/6$), confined between two plates in $(3 + 1)$-D flat space-time. Let the two plates be localized at $x_3 = 0$ and $x_3 = a > 0$, in Cartesian coordinates $\{t, x_1, x_2, x_3\}$. We impose Dirichlet boundary conditions on the plates:

$$\varphi(t, x_1, x_2, x_3 = 0) = \varphi(t, x_1, x_2, x_3 = a) = 0.$$

A complete set of mode solutions satisfying these boundary conditions is given by

$$u_{\vec{k}}(x) = \frac{1}{\sqrt{4\pi^2\omega_0}} e^{-i\omega t + k_1 x_1 + k_2 x_2} \sin(k_3 x_3),$$

where $k_3 = \frac{n\pi}{a}$, for $n \in \mathbb{N}$, $\vec{k} = (k_1, k_2, k_3)$ and $\omega^2 = |\vec{k}|^2$. The field can then be expanded in terms of these modes as

$$\hat{\varphi}(x) = \int_{\mathbb{R}^2} dk_1 dk_2 \sum_{n=1}^{\infty} \left( \hat{a}_{\vec{k}} u_{\vec{k}}(x) + \hat{a}_{\vec{k}}^{\dagger} u_{\vec{k}}^*(x) \right).$$
Then, the so-called Casimir vacuum $|c\rangle$ can be defined by
\[ \hat{a}_c^\dagger |c\rangle = 0, \quad \forall k_1, k_2 \in \mathbb{R}, n \in \mathbb{N}. \]

The Wightman function in the Casimir vacuum, $G^C_+(x, x') = \langle c|\hat{\phi}(x)\hat{\phi}(x')|c\rangle$, can be calculated via the method of images, giving
\[ G^C_+(x, x') = \sum_{n=-\infty}^{\infty} [G^M_+(t, \vec{x}; t', x'_1, x'_2, x'_3 + 2na) - G^M_+(t, \vec{x}; t', x'_1, x'_2, -x'_3 + 2na)], \]
where $G^M_+(x, x')$ is the Wightman function in the Minkowski vacuum, given by eq.(2).

(a) Using the point-splitting method, obtain the following expression for the renormalized expectation value of $\hat{\phi}^2(x)$ in the Casimir vacuum:
\[ \langle c| : \hat{\phi}^2(x) : |c\rangle = \lim_{x \to x'} \left[ G^C_+(x, x') - G^M_+(x, x') \right] = \frac{1}{48a^2} \left( 1 - 3 \csc^2 \left( \frac{\pi x_3}{a} \right) \right). \]

Hint: use the following series,
\[ \csc^2(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{(\xi - n\pi)^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \]

(b) Use the results from part (a) to obtain the following expression for the renormalized expectation value of the stress-energy tensor in the Casimir vacuum:
\[ \langle c| : \hat{T}^{\mu \nu} : |c\rangle = \frac{\pi^2}{1440a^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

You may use the expression for the stress-energy tensor of a real, massless, conformally coupled scalar field in flat space-time:
\[ T^{\mu \nu} = \frac{2}{3} \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{6} \varphi \partial^\mu \partial^\nu \varphi + \eta^{\mu \nu} \left( \frac{1}{3} \varphi \Box \varphi - \frac{1}{6} \partial_\mu \varphi \partial^\mu \varphi \right). \]

You may also use the point-splitting method:
\[ \langle c| : \left( \frac{\partial \hat{\varphi}}{\partial t} \right)^2 : |c\rangle = \lim_{x \to x'} \frac{\partial^2}{\partial t \partial t'} \left[ G^C_+(x, x') - G^M_+(x, x') \right]. \]