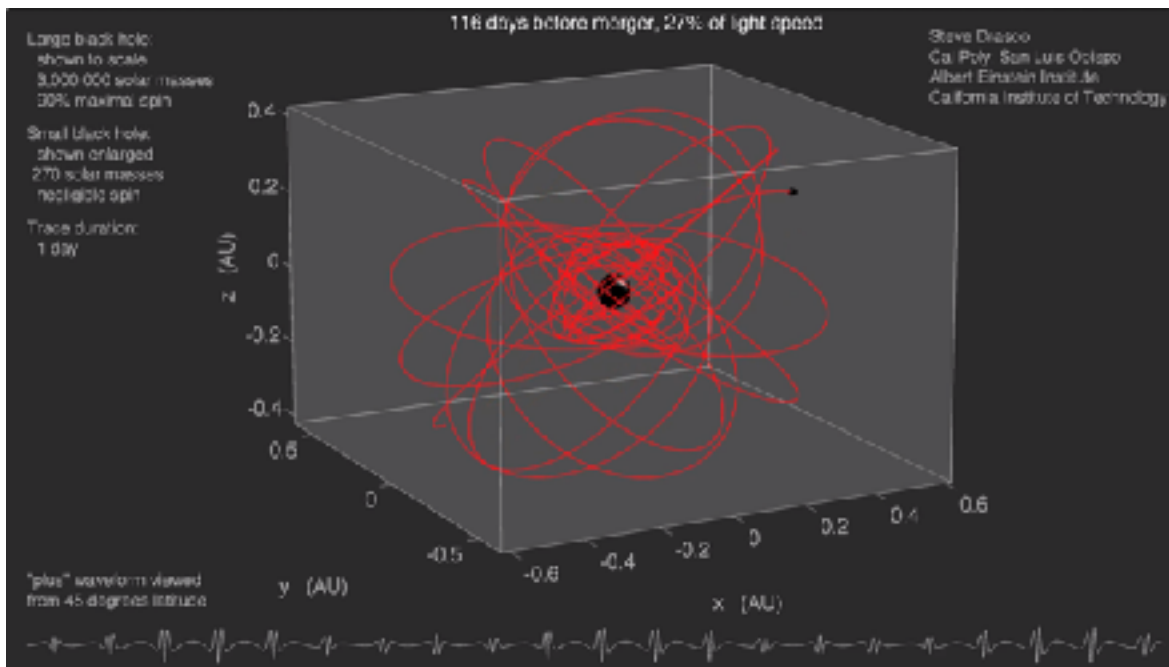


“Fairly fast” adiabatic EMRI waveforms



Progress toward
making “fast”
Teukolsky-based
EMRI waveforms

Based on ongoing work with Niels Warburton
with critical contributions from Gaurav Khanna

Review: Equations of motion

Flanagan and Hinderer [PRL **109**, 071102 (2012)] described motion in action-angle framework:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

$$\frac{dJ_i}{d\lambda} = \varepsilon G_i^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

λ is a time variable (often Mino time for Kerr),
and $\varepsilon = m/M$. Action and angle variables are

$$q_\alpha = (q_t, q_r, q_\theta, q_\phi) \quad J_i = (E/m, L_z/m, Q/m^2)$$

Generalized angle variables
describing motion in
spacetime coordinates

Integrals of motion, describe
background motion.

Review: Equations of motion

Consider these equations order-by-order to understand motion that follows in this framework:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

$$\frac{dJ_i}{d\lambda} = \varepsilon G_i^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

Review: Equations of motion

Consider these equations order-by-order to understand motion that follows in this framework:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J})$$

At order ε^0

$$\frac{dJ_i}{d\lambda} = 0$$

Zeroth order in mass ratio: Generalized angle variables accumulate at rate set by their frequency; integrals of motion are conserved.

In simpler words: The zeroth order motion is a background spacetime geodesic.

Review: Equations of motion

Consider these equations order-by-order to understand motion that follows in this framework:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J})$$

At order ε^1

$$\frac{dJ_i}{d\lambda} = \varepsilon G_i^{(1)}(q_\theta, q_r, \mathbf{J})$$

Very useful to understand these additional “forcing” terms by considering their Fourier expansions:

$$F(q_\theta, q_r) = \sum_{k,n} F_{kn} e^{i(kq_\theta + nq_r)}$$

$$F_{kn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dq_\theta \int_0^{2\pi} dq_r F(q_\theta, q_r) e^{-i(kq_\theta + nq_r)}$$

Two timescales

Fourier expansion lets us identify terms that oscillate on short timescales and terms that accumulate secularly over longer intervals. For most orbits,

$$F(q_\theta, q_r) = \langle F \rangle + \delta F(q_\theta, q_r)$$

where

$$\langle F \rangle = F_{00}$$



Accumulates over
many orbits

$$\delta F = \sum_{k,n \neq 0} F_{kn} e^{i(kq_\theta + nq_r)}$$



Oscillates over an orbit,
averages to (nearly) zero

Two timescales

The equations of motion then become

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha + \varepsilon \langle g_\alpha^{(1)} \rangle + \varepsilon \delta g_\alpha^{(1)}$$

$$\frac{dJ_i}{d\lambda} = \varepsilon \langle G_i^{(1)} \rangle + \varepsilon \delta G_i^{(1)}$$

Average of the forcing function $G^{(1)}_i$ describes the leading evolution of integrals of motion:

$$\langle G_i^{(1)}(\mathbf{J}) \rangle = \left[\left\langle \frac{dE}{d\lambda} \right\rangle, \left\langle \frac{dL_z}{d\lambda} \right\rangle, \left\langle \frac{dQ}{d\lambda} \right\rangle \right]$$

Drives secular evolution of the system's orbital parameters, on time scale $M/\varepsilon = M^2/m$.

Two timescales

The equations of motion then become

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha + \varepsilon \langle g_\alpha^{(1)} \rangle + \varepsilon \delta g_\alpha^{(1)}$$

$$\frac{dJ_i}{d\lambda} = \varepsilon \langle G_i^{(1)} \rangle + \varepsilon \delta G_i^{(1)}$$

Average of the forcing function $g^{(1)}_a$ equivalent to a shift in the orbital frequencies:

$$\omega_\alpha(\mathbf{J}) \rightarrow \omega_\alpha(\mathbf{J}) + \varepsilon \langle g_\alpha^{(1)}(\mathbf{J}) \rangle$$

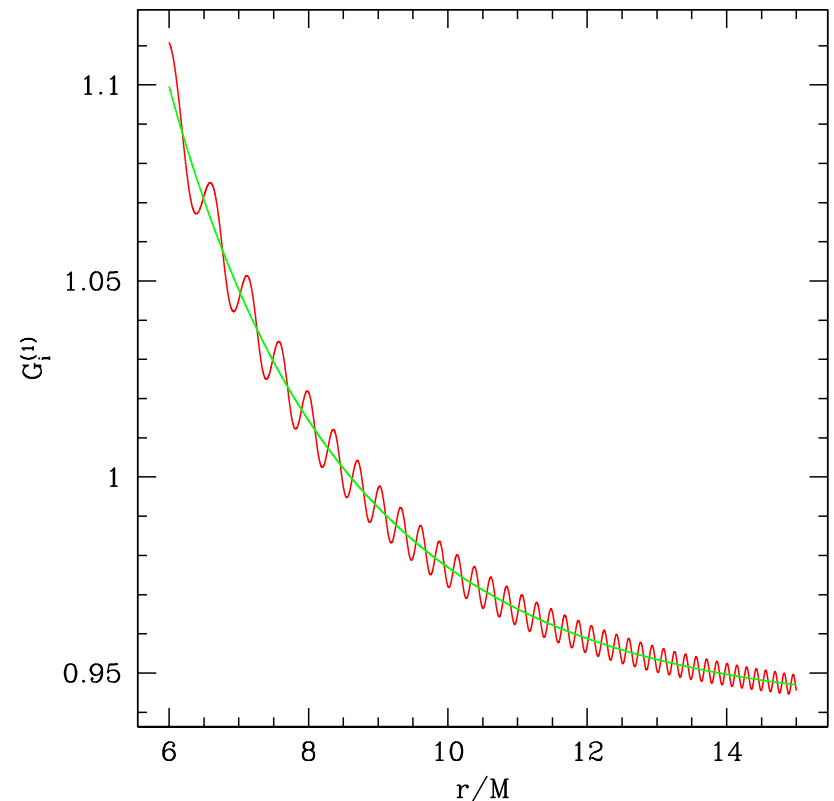
Leading conservative impact on the binary:
Frequencies are shifted of order $\varepsilon = m/M$.

Adiabatic approximation

Throw out oscillatory parts, integrate up motion with bits that survive long-time averaging.

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon \langle g_\alpha^{(1)}(\mathbf{J}) \rangle \quad \frac{dJ_i}{d\lambda} = \varepsilon \langle G_i^{(1)}(\mathbf{J}) \rangle$$

This averaging is expected to work well for “most” orbits because the true behavior tracks the averaged behavior, modulo short timescale oscillations which have subdominant impact.

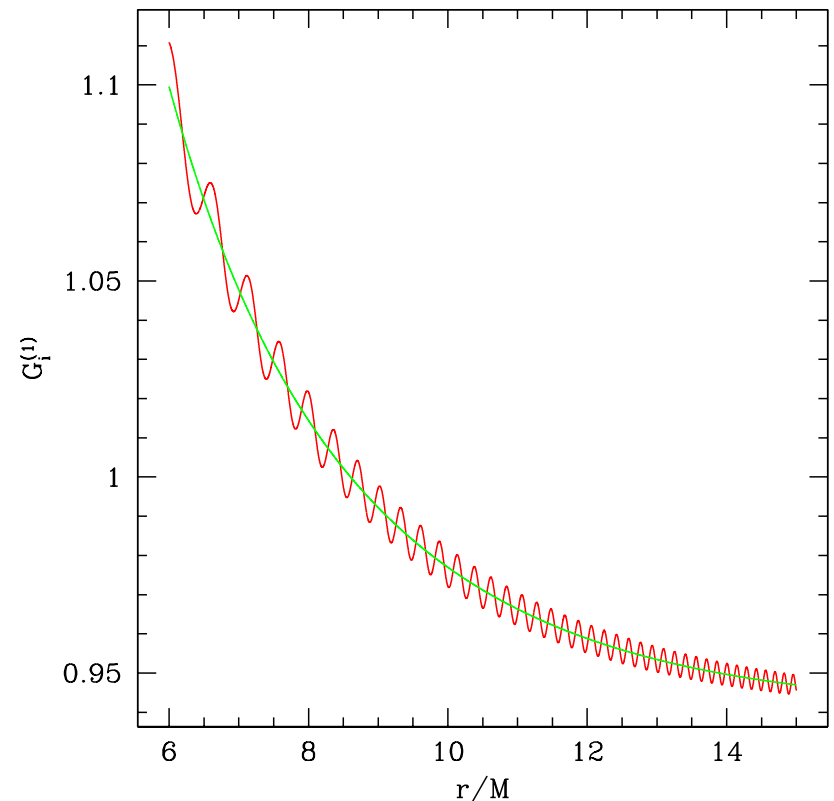


Adiabatic approximation

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$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon \langle g_\alpha^{(1)}(\mathbf{J}) \rangle \quad \frac{dJ_i}{d\lambda} = \varepsilon \langle G_i^{(1)}(\mathbf{J}) \rangle$$

Fails at resonances!
(Topic for different discussion.)



Radiative approximation

Not uncommon to neglect leading conservative correction, modeling system as

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) \qquad \frac{dJ_i}{d\lambda} = \varepsilon \langle G_i^{(1)}(\mathbf{J}) \rangle$$

Can be justified: leading conservative correction to $dq_a/d\lambda$ has the same magnitude of effect as the leading “post-adiabatic” oscillatory piece of $dJ_i/d\lambda$.

HOWEVER, reason that these terms contribute at this level arises from different physics.
Worth keeping nomenclature distinct!

Radiative approximation

Not uncommon to neglect leading conservative correction, modeling system as

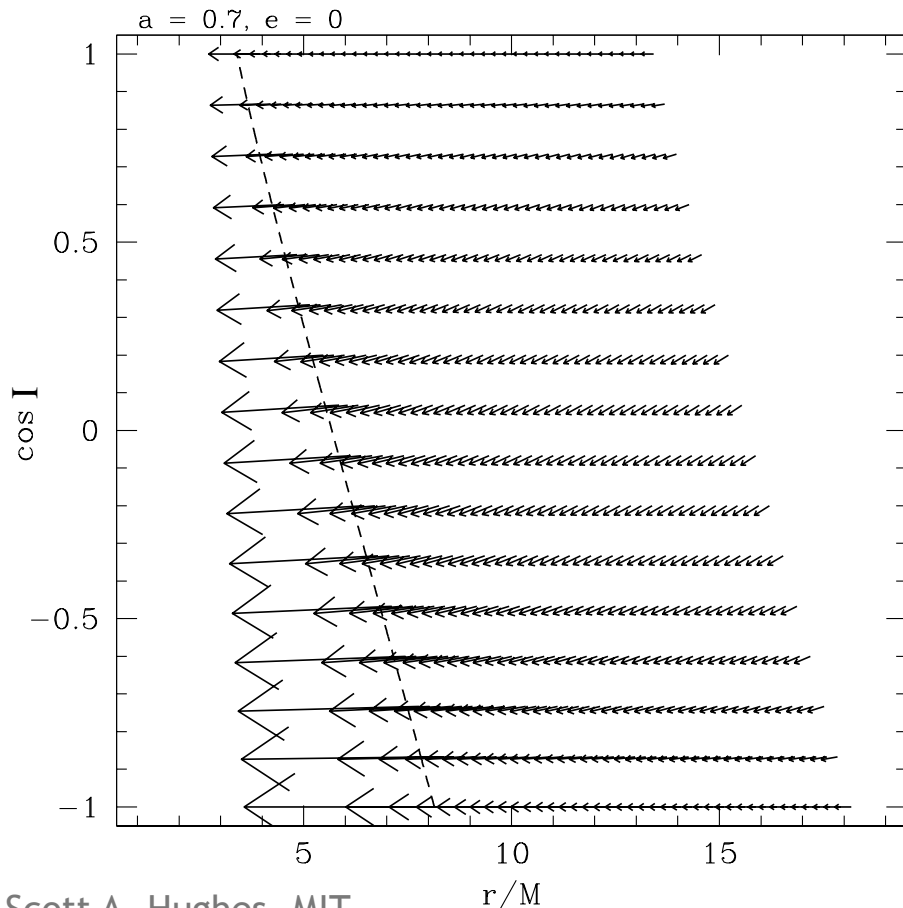
$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) \qquad \frac{dJ_i}{d\lambda} = \varepsilon \langle G_i^{(1)}(\mathbf{J}) \rangle$$

In this approach, inspiral is approximated as a sequence of geodesics ... sequence is determined by computing the “fluxes” of orbit integrals (E, L_z, Q) .

Resulting waveforms are what we mean by “Teukolsky-based” EMRI waveform models.

Making waveforms

To make time-domain Teukolsky-based waveforms, compute fluxes on a grid of geodesics. Move along “flow lines” defined by fluxes of integrals of motion; compute amplitude and phase along this flow line.

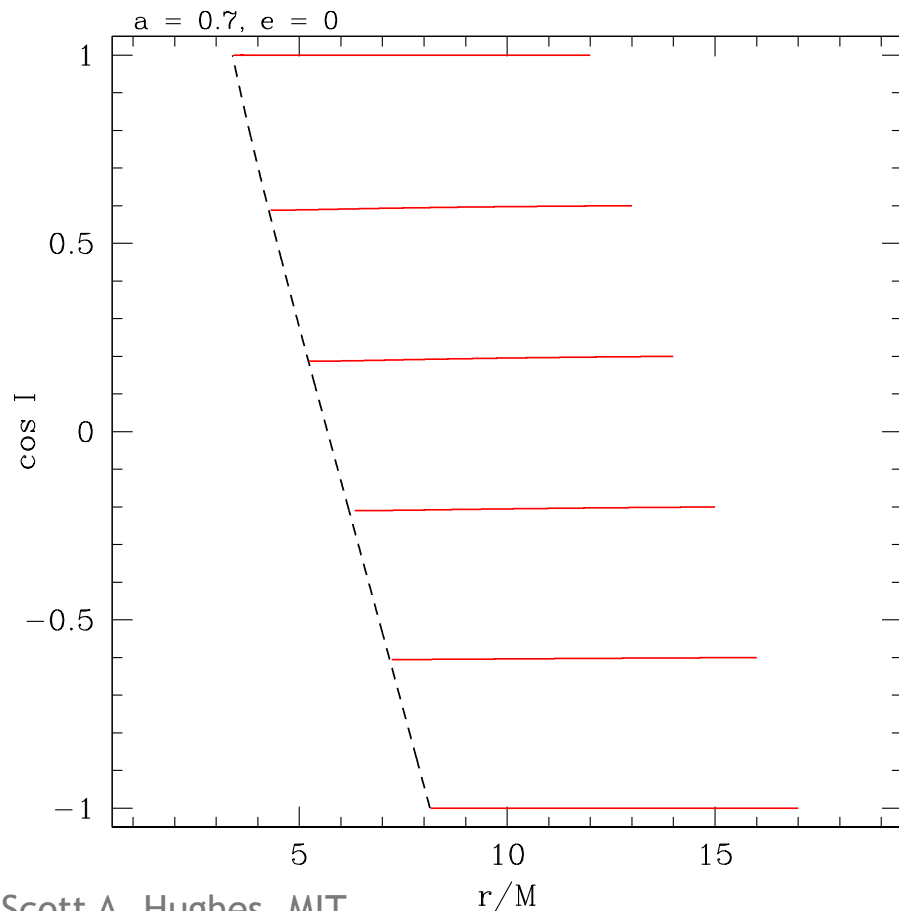


Example: Tail of each arrow is a particular circular orbit in the $(r, \cos I)$ plane.

Head of arrow is direction adiabatic radiation pushes on orbit; length is proportional to how strongly it pushes.

Making waveforms

To make time-domain Teukolsky-based waveforms, compute fluxes on a grid of geodesics. Move along “flow lines” defined by fluxes of integrals of motion; compute amplitude and phase along this flow line.



Example: Tail of each arrow is a particular circular orbit in the $(r, \cos I)$ plane.

Result for this example: Inspiral flows on curves of nearly constant $\cos I$.

Making waveforms

To make time-domain Teukolsky-based waveforms, compute fluxes on a grid of geodesics. Move along “flow lines” defined by fluxes of integrals of motion; compute amplitude and phase along this flow line.

$$h(t; \vartheta, \varphi) = \frac{1}{r} \sum_{lmkn} H_{lmkn} S_{lm}[\vartheta; a\omega_{mkn}(t)] e^{i[m\varphi - \Phi_{mkn}(t)]}$$

$$H_{lmkn} = -2Z_{lmkn}/\omega_{mkn}^2 \quad \Phi_{mkn}(t) = \int_{t_0}^t \omega_{mkn}(t') dt'$$

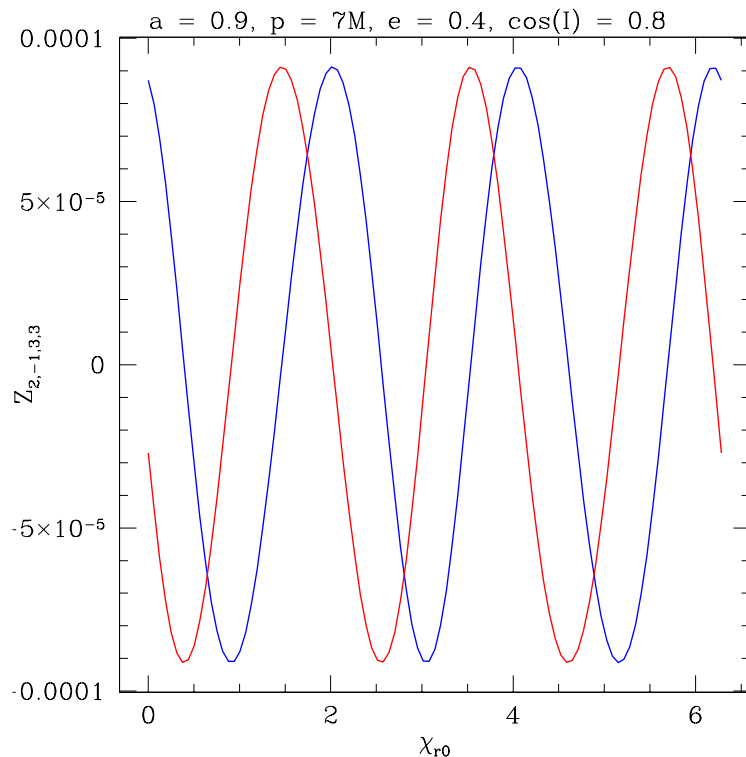
**Some of these quantities are expensive ...
but they can be computed in advance,
stored, and provided for general use.**

Phase of Z_{lmkn} / H_{lmkn}

The complex amplitude Z_{lmkn} (found by solving the radial Teukolsky equation) depends upon the relative phase of the r , θ , and φ motions.

$$r = \frac{p}{1 + e \cos(\chi_r + \chi_{r0})}$$

$$\cos \theta = \sin I \cos(\chi_\theta + \chi_{\theta 0})$$



Example of Z_{lmkn} as a function of “initial” radial phase χ_{r0} (blue = real, red = imag)

Phase of Z_{lmkn} / H_{lmkn}

Phase of the amplitude depends in a simple way on an orbit's initial condition parameters:

$$Z_{lmkn} = e^{i\xi_{mkn}} \check{Z}_{lmkn}$$

where $\check{Z}_{lmkn}$ is the amplitude of the “*fiducial geodesic*,” the geodesic for which all the offset phases are zero when clocks start ticking, and where

$$\begin{aligned} \xi_{mkn} = & m \left[\Delta\check{\phi}_r(\chi_{r0}) + \Delta\check{\phi}_\theta(\chi_{\theta0}) - \phi_0 \right] \\ & + k\Upsilon_\theta \lambda_{\theta0}(\chi_{\theta0}) \\ & + n\Upsilon_r \lambda_{r0}(\chi_{r0}) \\ & - \omega_{mkn} \left[\Delta\check{t}_r(\chi_{r0}) + \Delta\check{t}_\theta(\chi_{\theta0}) \right] \end{aligned}$$

(Derivation: Drasco, Flanagan, Hughes 2005)

Quantities used to describe φ and t as functions of λ
(Drasco & Hughes 2004; Fujita & Hikida 2009)

Phase of Z_{lmkn} / H_{lmkn}

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Comment 1: χ_{r0} and $\chi_{\theta0}$ are constant along inspiral if only dissipative effects are considered.

(Derivation: Drasco, Flanagan, Hughes 2005)

Phase of Z_{lmkn} / H_{lmkn}

Phase of the amplitude depends in a simple way on an orbit's initial condition parameters:

$$Z_{lmkn} = e^{i\xi_{mkn}} \check{Z}_{lmkn}$$

where $\check{Z}_{lmkn}$ is the amplitude of the “*fiducial geodesic*,” the geodesic for which all the offset phases are zero when clocks start ticking, and where

Comment 2: This phase can be written

$$\xi_{mkn} = m\xi_{100} + k\xi_{010} + n\xi_{001}$$

significantly reducing what must be computed / stored.

Complicated waveform = Sum over simple “voices”

$$h(t; \vartheta, \varphi) = \frac{1}{r} \sum_{lmkn} H_{lmkn} S_{lm}[\vartheta; a\omega_{mkn}(t)] e^{i[m\varphi - \Phi_{mkn}(t)]}$$

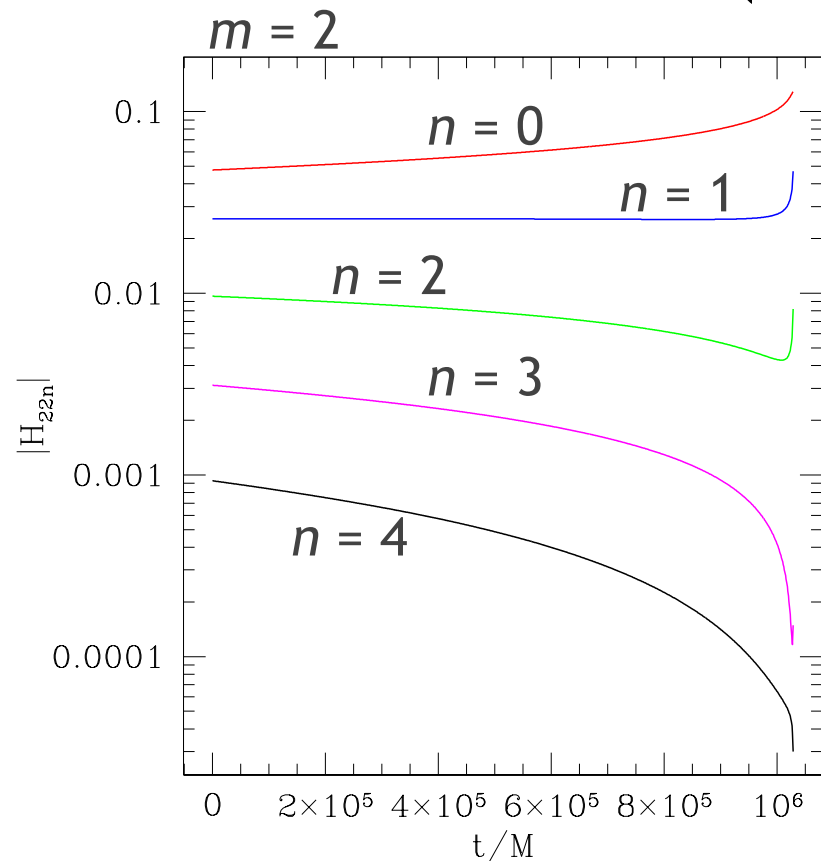
Can be very
complicated

Each term is quite
smooth and simple

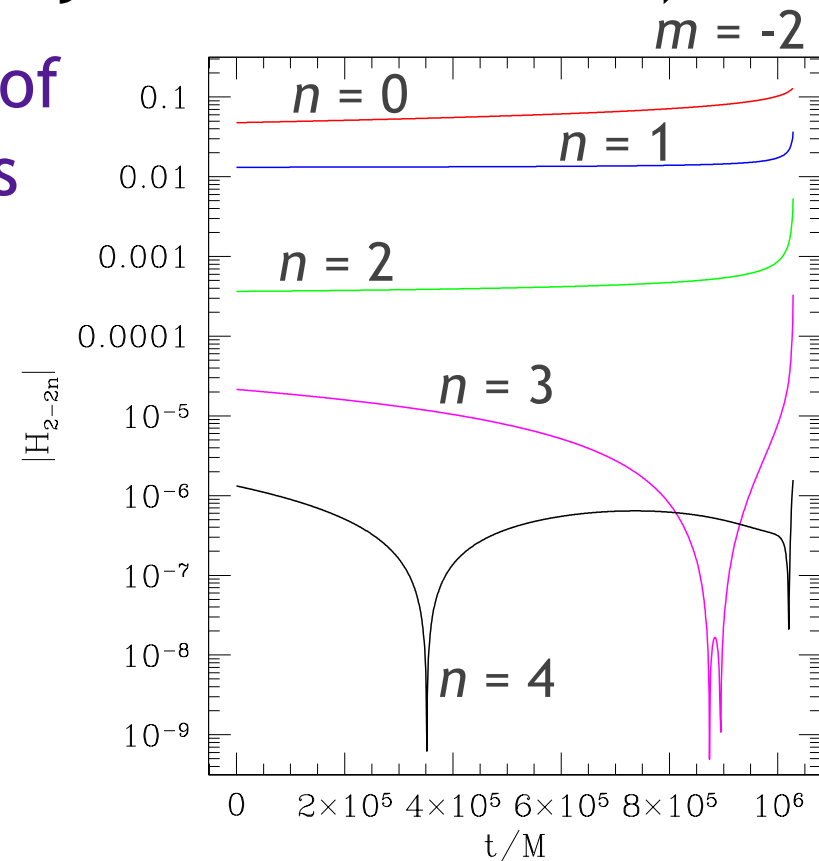
“Voices” which contribute to a radiative EMRI
model can be sampled much more sparsely
than the total waveform suggests.

1st test case: Low eccentricity inspiral into Schwarzschild

With Niels Warburton, have studied inspiral that begins with $p = 16M$, $e = 0.2$ into $a = 0$; compare with time-domain solution (data provided by Gaurav Khanna).

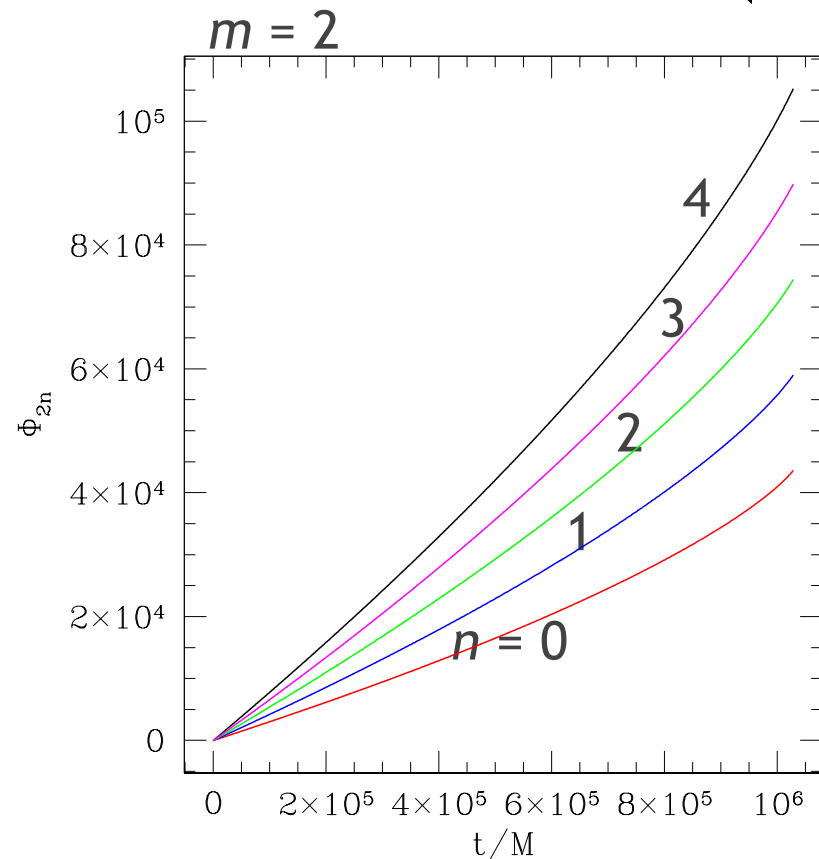


Amplitudes of
 $l = 2$ voices



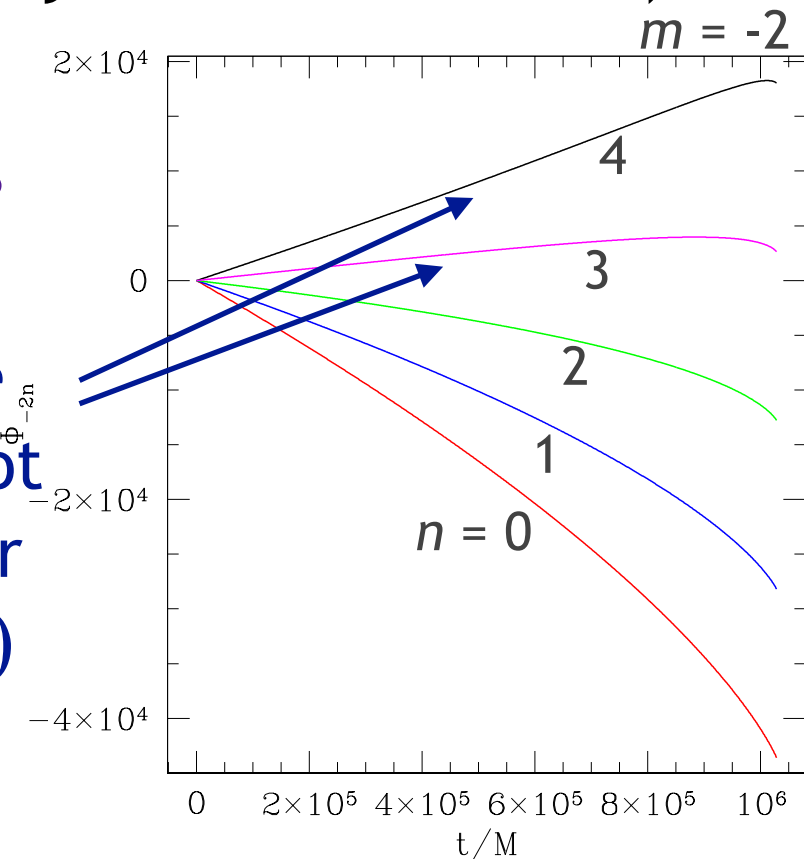
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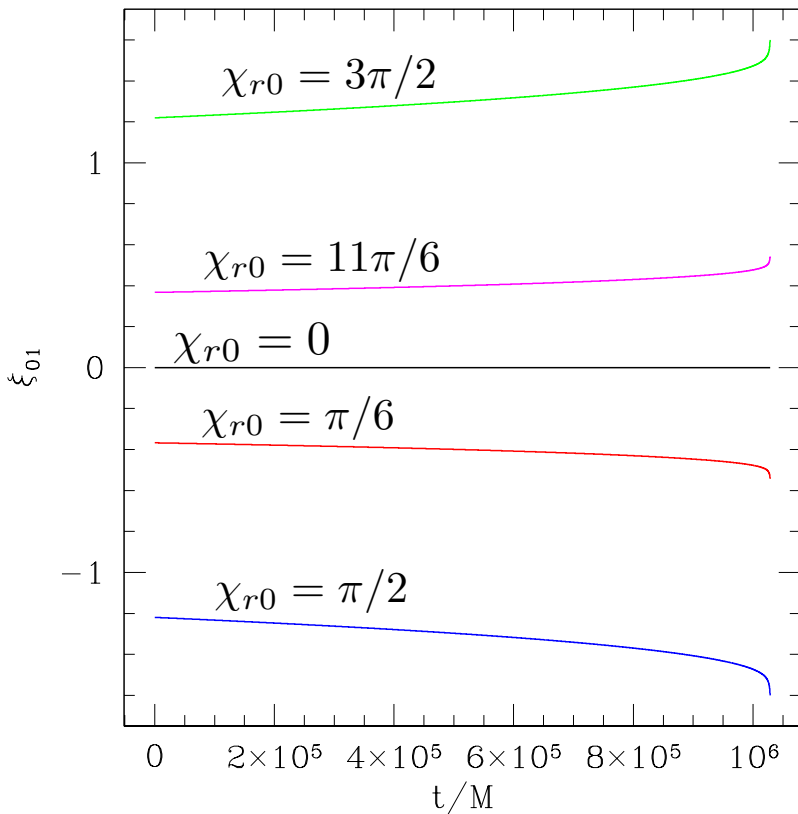
Phases of
 $l = 2$ voices

Notice phase
evolution is not
monotonic for
 $(m,n) = (-2,3)$
and $(-2,4)$



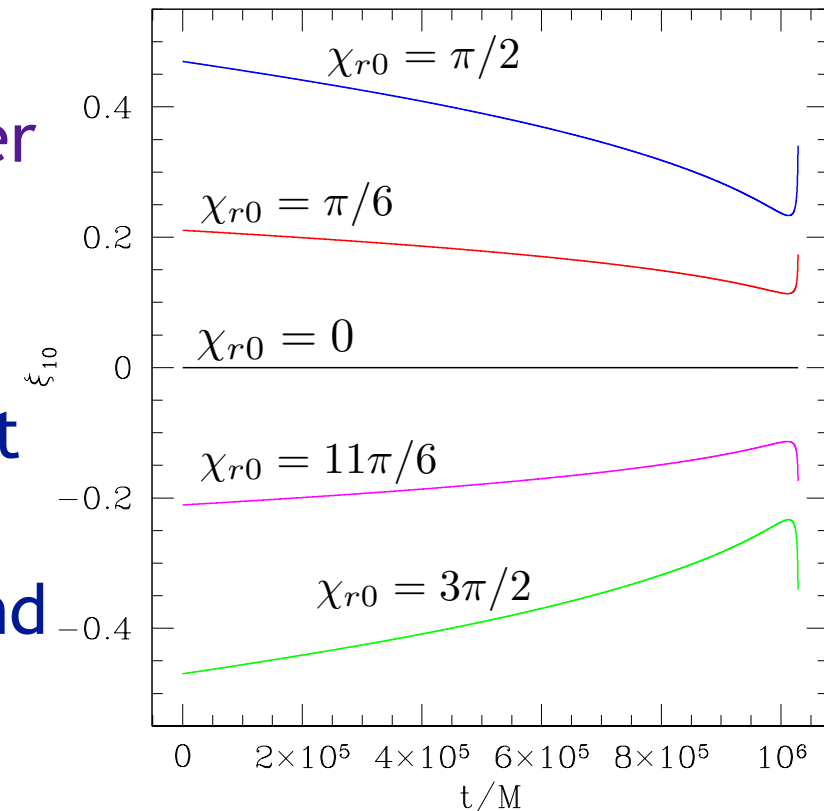
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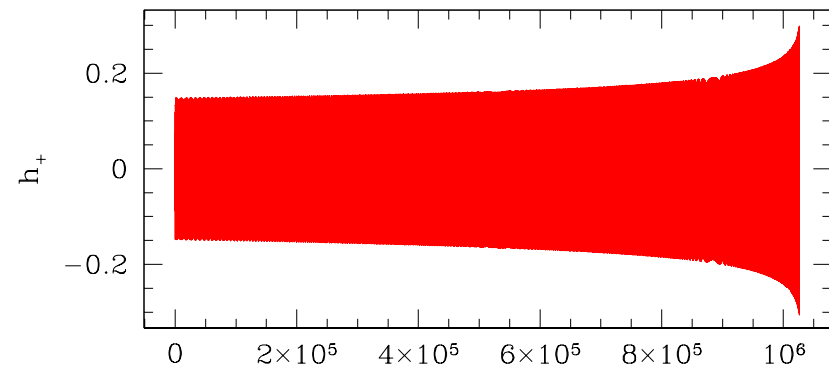
Behavior of phase ξ_{mn} over inspiral.

Note: χ_{r0} is constant, but mapping between it and ξ_{mn} evolves.

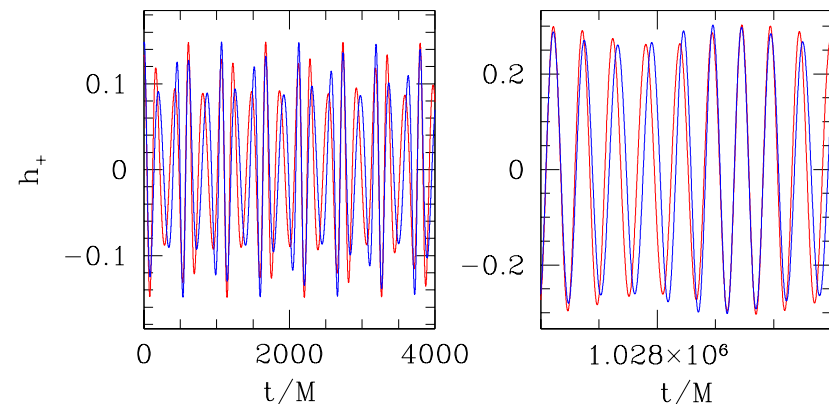


1st test case: Low eccentricity inspiral into Schwarzschild

With Niels Warburton, have studied inspiral that begins with $p = 16M$, $e = 0.2$ into $a = 0$; compare with time-domain solution (data provided by Gaurav Khanna).



Time-domain waveform we assemble (mass ratio $\eta = 10^{-4}$) ... general behavior at top, impact of phase correction ξ_{mn} on bottom.

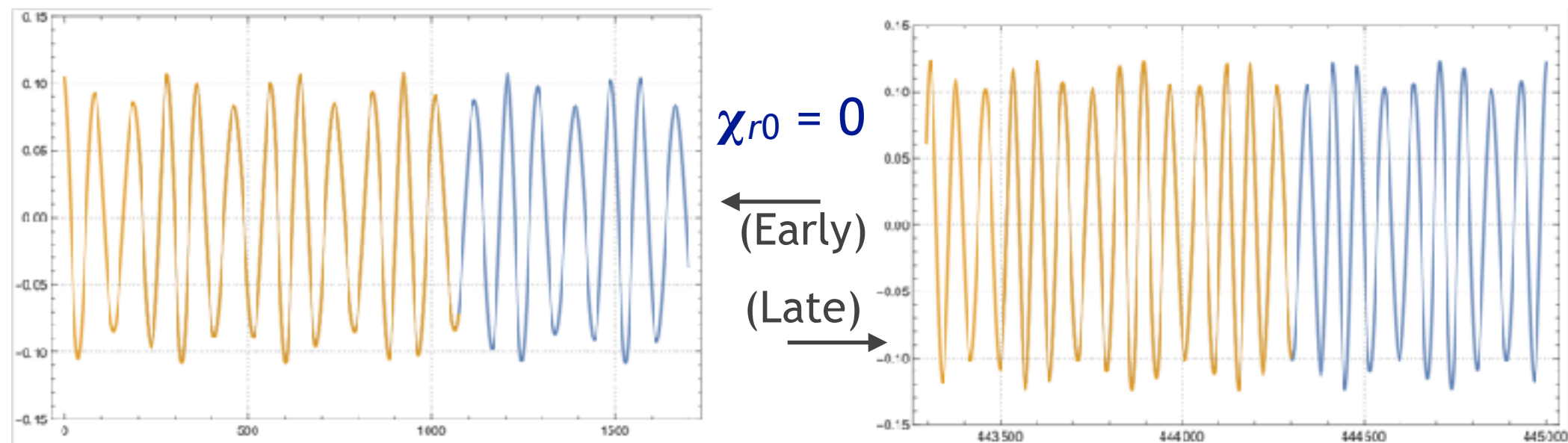


Red curves: radial phase $\chi_{r0} = \pi/3$

Blue curves: radial phase $\chi_{r0} = 0$

1st test case: Low eccentricity inspiral into Schwarzschild

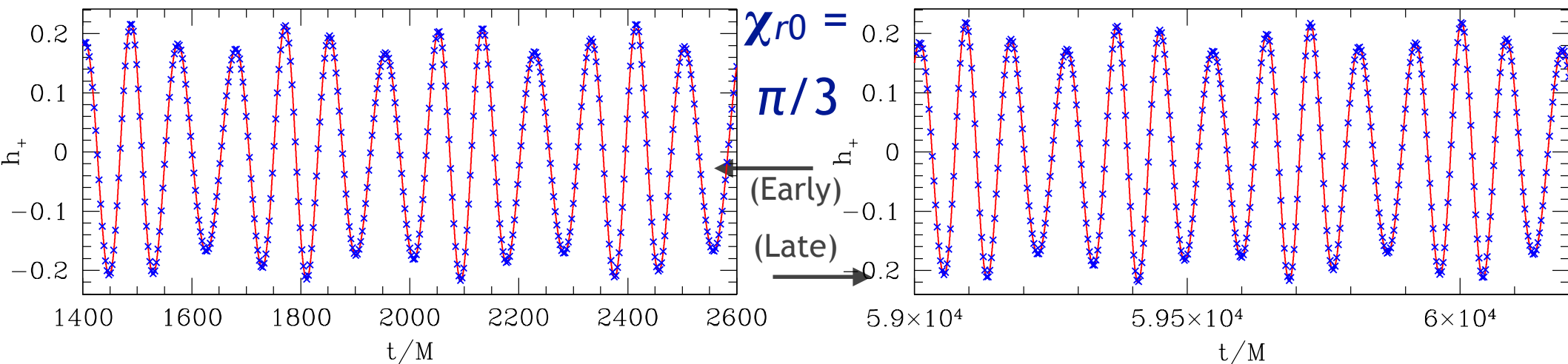
With Niels Warburton, have studied inspiral that begins with $p = 16M$, $e = 0.2$ into $a = 0$; compare with time-domain solution (data provided by Gaurav Khanna).



Blue: Time domain waveform computed by Khanna
Yellow: Frequency domain waveform

1st test case: Low eccentricity inspiral into Schwarzschild

With Niels Warburton, have studied inspiral that begins with $p = 16M$, $e = 0.2$ into $a = 0$; compare with time-domain solution (data provided by Gaurav Khanna).



Blue xs: Time domain waveform computed by Khanna
Red curve: Frequency domain waveform

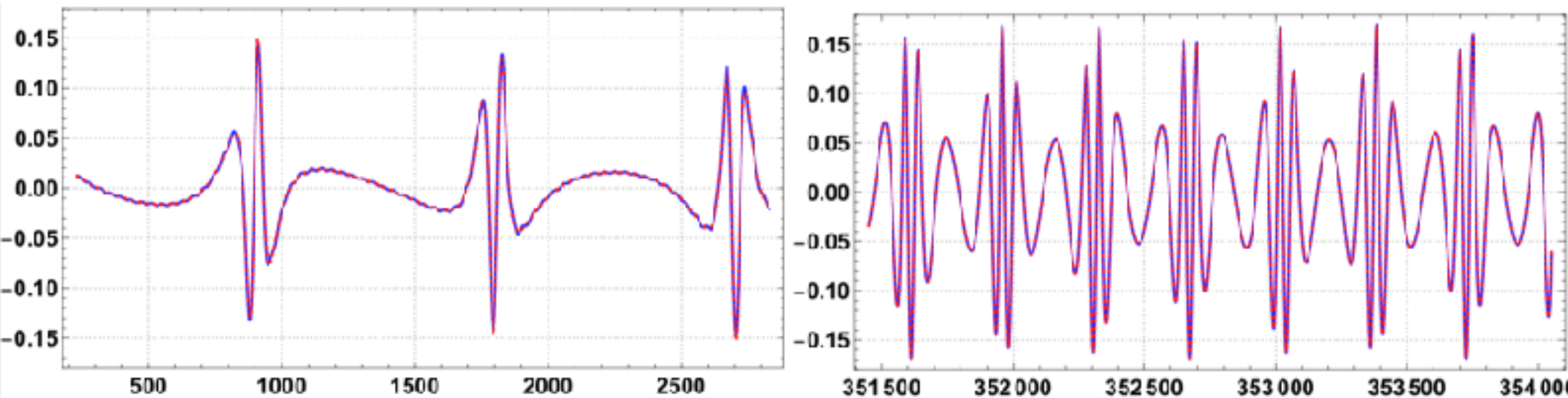
In progress: High eccentricity into Schwarzschild

Second case we've looked at briefly: inspiral that begins with $p = 12.5M$, $e = 0.7$ into $a = 0$; compare with time-domain solution (data provided by Gaurav Khanna).

(Early)

$$\chi_{r0} = 0$$

(Late)



Red curve: Time domain waveform computed by Khanna
Blue curve: Frequency domain waveform

Frequency domain

At Capra 21, Stas Babak exhorted us to make frequency-domain EMRI waveforms ... especially important as these models get linked to science and data analysis studies.

Seems to be a natural place to use the *stationary phase approximation (SPA)*: given

$$h(t) = A(t)e^{-i\Phi(t)} , \quad F(t) \equiv \frac{1}{2\pi} \frac{d\Phi}{dt} , \quad \dot{F}(t) \equiv \frac{1}{2\pi} \frac{d^2\Phi}{dt^2}$$

and given

$$\frac{1}{F} \frac{dA}{dt} \ll A , \quad \frac{\dot{F}}{F} \ll F$$

Then:

$$\tilde{h}(f) \simeq \frac{A[t(f)]e^{-i[\pi/4+\Phi_0]}}{\sqrt{\dot{F}[t(f)]}}$$

where $t(f)$ is the time at which $F = f$.

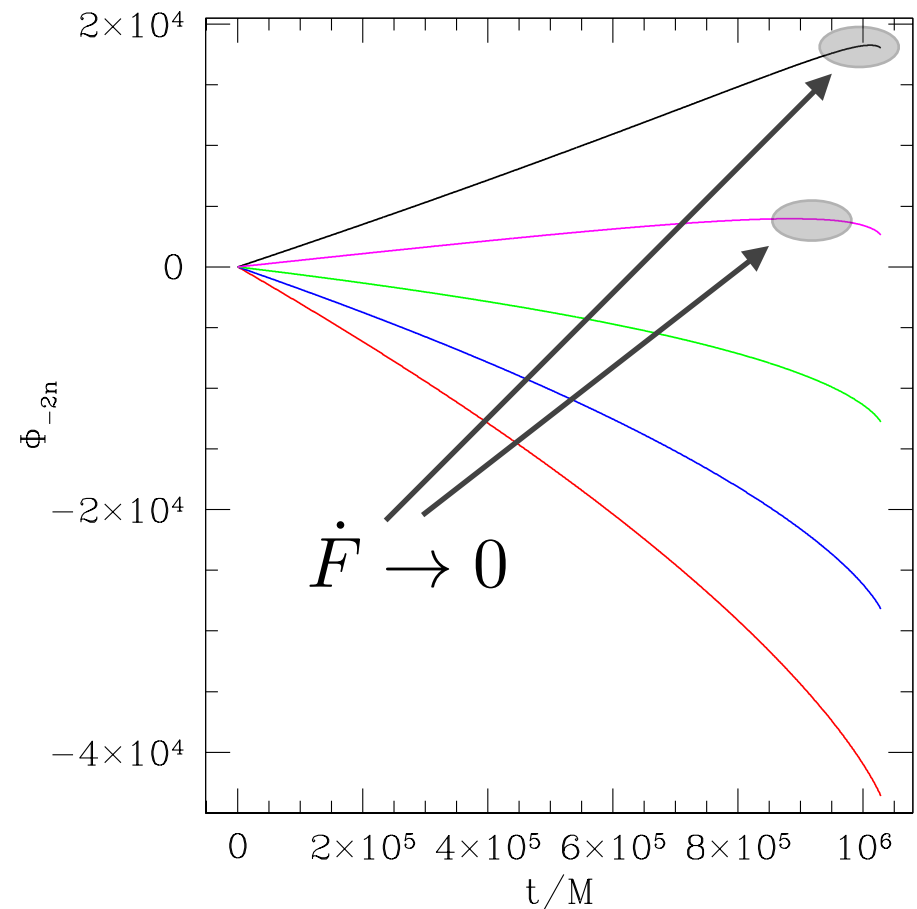
Breaks down for some voices

Simple in principle to extend this to a signal with multiple voices ... but run into a problem with voices for which phase evolution is not monotonic.

Saw an example of this behavior in the example shown a few slides ago:

$$\tilde{h}(f) \simeq \frac{A[t(f)]e^{-i[\pi/4+\Phi_0]}}{\sqrt{\dot{F}[t(f)]}}$$

The SPA formula diverges here!

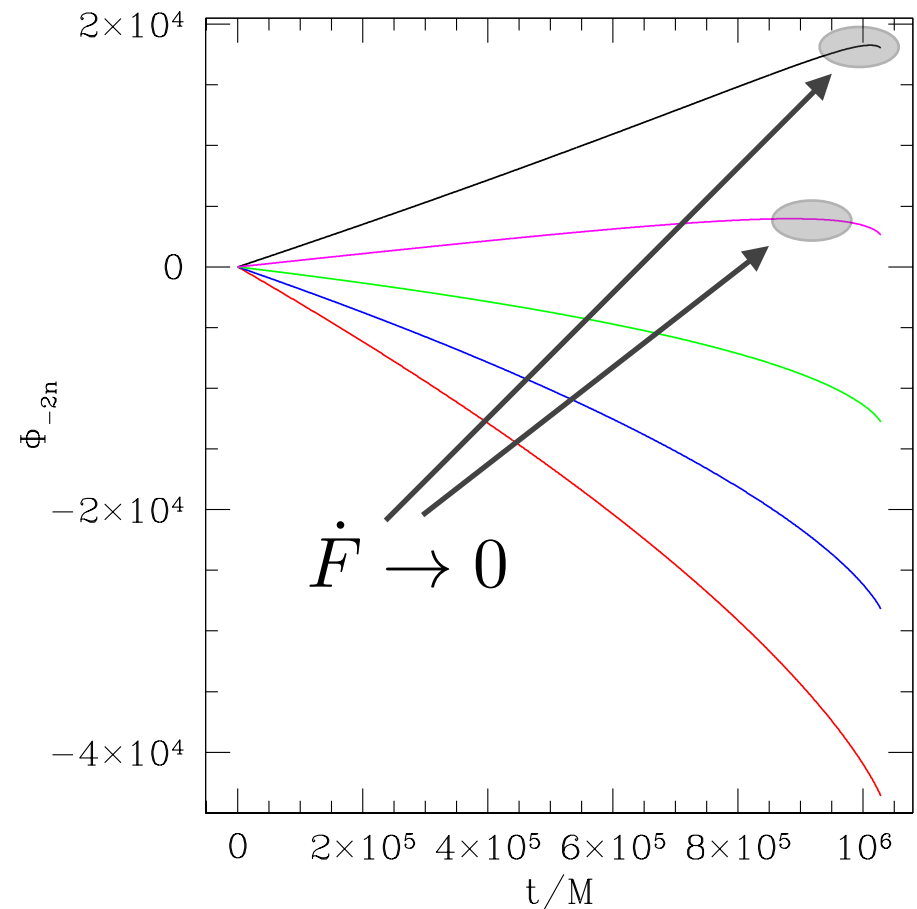


Breaks down for some voices

Simple in principle to extend this to a signal with multiple voices ... but run into a problem with voices for which phase evolution is not monotonic.

Reason is that SPA assumes instantaneous frequency and frequency's rate of change completely characterizes $\Phi(t)$.

Need to include more information about how the phase is evolving.



Extended SPA

Redo the stationary phase calculation, but include information about the next term: $2\pi\ddot{F} = d^3\Phi/dt^3$.

$$\begin{aligned}\tilde{h}(f) &= \int_{-\infty}^{\infty} A(t) e^{i[2\pi ft - \Phi(t)]} dt \\ &\simeq \int_{-\infty}^{\infty} A(t) e^{i[2\pi ft - \Phi_0 - 2\pi Ft - \pi \dot{F}t^2 - \pi \ddot{F}t^3/3]} dt\end{aligned}$$

Standard calc: Neglect $\ddot{F}$ term, note integrand oscillates least when $f = F$ (i.e., phase is stationary).

Extended: Repeat, but include $\ddot{F}$:

$$\tilde{h}(f) \simeq \lim_{\gamma \rightarrow 0} A[t(f)] e^{-i\Phi_0} \exp \left[\frac{(\gamma + 2\pi i \dot{F})^3}{12\pi^2 \ddot{F}^2} \right] K_{1/3} \left[\frac{(\gamma + 2\pi i \dot{F})^3}{12\pi^2 \ddot{F}^2} \right]$$

Extended SPA

Redo the stationary phase calculation, but include information about the next term: $2\pi\ddot{F} = d^3\Phi/dt^3$.

$$\begin{aligned}\tilde{h}(f) &= \int_{-\infty}^{\infty} A(t) e^{i[2\pi ft - \Phi(t)]} dt \\ &\simeq \int_{-\infty}^{\infty} A(t) e^{i[2\pi ft - \Phi_0 - 2\pi Ft - \pi \dot{F} t^2 - \pi \ddot{F} t^3 / 3]} dt\end{aligned}$$

Note: when $\ddot{F}$ is “small,” not too hard to show that this yields a simple correction to standard SPA:

$$\tilde{h}(f) \simeq \frac{A[t(f)] e^{-i(\Phi_0 + \pi/4)}}{\sqrt{\dot{F}[t(f)]}} \left(1 + \frac{5i}{48\pi} \frac{(\ddot{F})^2}{(\dot{F})^3} \right)$$

Moving forward

Planned next steps:

1. More example cases; careful profiling of CPU time for making waveforms.
2. Data sets for equatorial orbits and circular orbits already exist, will be released through Black Hole Perturbation Theory Toolkit very soon (some data already there).
 - * One partial generic data set exists; porting GREMLIN to a larger compute cluster to expand.
3. Software for processing data, generating waveforms will be provided with Teukolsky and flux data.
4. **Frequency domain!** Have not yet tested or implemented any of that yet.

Description of available data sets

Equatorial eccentric

Sets exist for $a/M = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99$; prograde and retrograde.

Has $e = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$.

Evenly spaced in

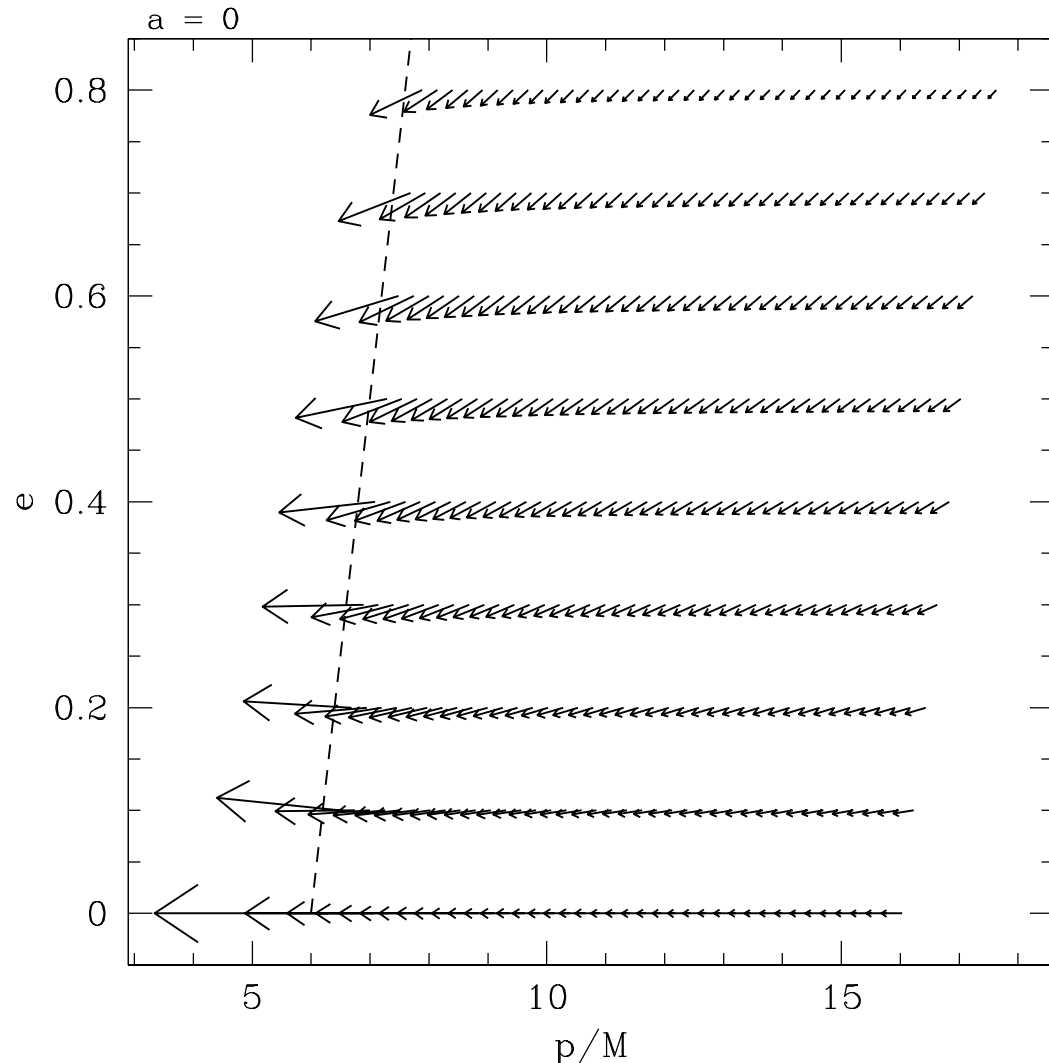
$$u = 1/\sqrt{p - 0.9p_{\text{LSO}}}$$

from $p_{\text{min}} = p_{\text{LSO}} + 0.02M$
to $p_{\text{max}} = p_{\text{min}} + 10M$.

All l from 2 to 10

All m for each l

Enough n for fluxes to
converge to 10^{-8} .



Circular inclined

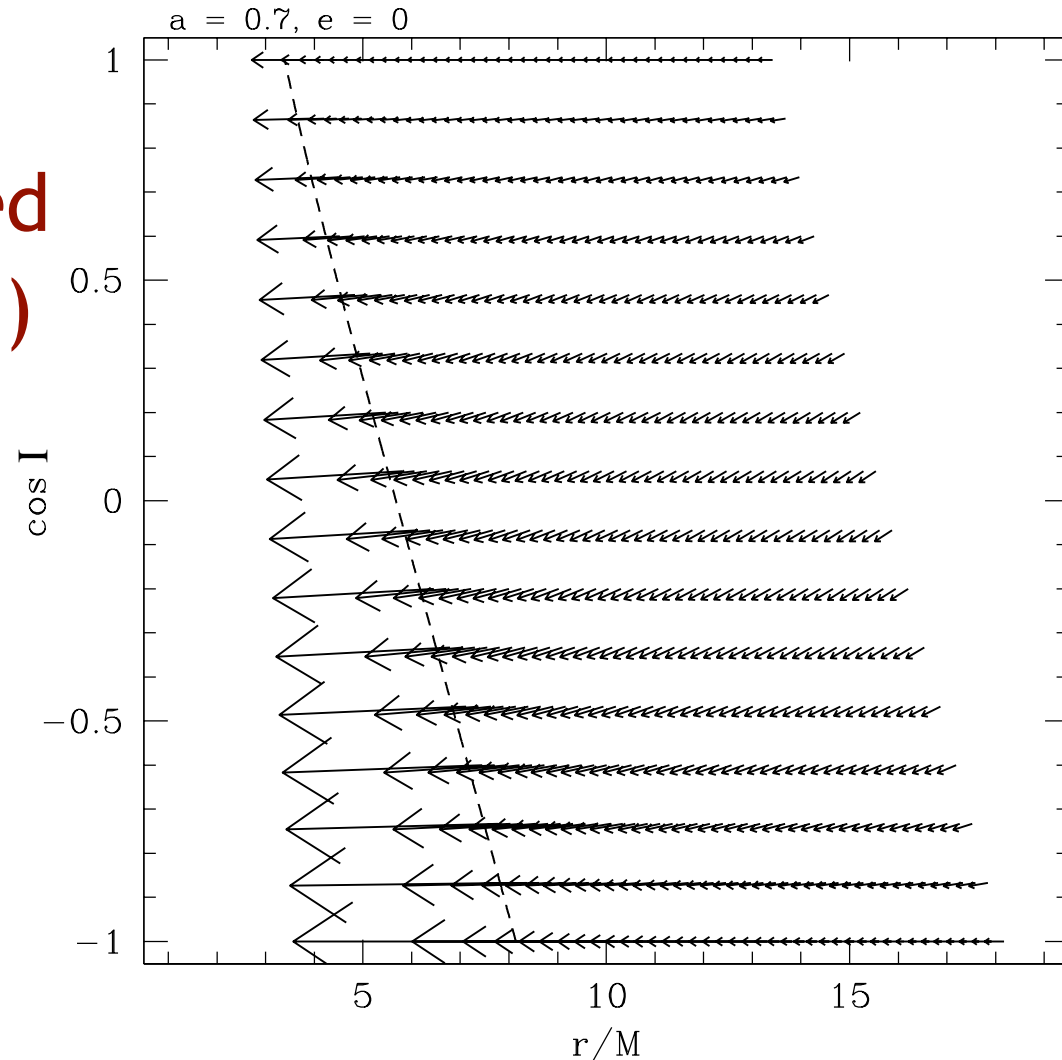
Same range of spin as equatorial data; spacing in r identical to the algorithm used to space in p .

For each spin, covers 16 values of $\cos i$, evenly spaced between prograde ($\cos i = 1$) and retrograde ($\cos i = -1$).

All l from 2 to 10

All m for each l

Enough k for fluxes to converge to 10^{-10} .



Generic

Only one case started so far: $a = 0.7M$

16 inclinations from prograde eq to retrograde eq

Eccentricities $e = 0, 0.1, 0.2, 0.3, 0.4$

p evenly spaced between $p_{\min} = p_{\text{LSO}} + 0.02M$
and $p_{\max} = p_{\min} + 10M$

Generic data sets have **MUCH** greater computational cost: Orbital parameter space is 3-D; each mode has 4 indices (spheroidal mode l and harmonics of all 3 orbital frequencies). Test case saturated easily available resources at MIT, ran for several weeks.