

Particle acceleration by large-scale turbulence

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Abstract: We investigate particle acceleration in large-scale turbulence with a length scale larger than the particle mean free path. We derive an ensemble-averaged transport equation of energetic charged particles from an extended transport equation that contains the shear acceleration. The ensemble-averaged transport equation describes particle acceleration by incompressible turbulence (turbulent shear acceleration). We find that for Kolmogorov turbulence, the turbulent shear acceleration becomes important in small scale. Furthermore, the turbulent shear acceleration could be more efficient than the second-order acceleration by Alfvén waves for super-Alfvénic turbulence.

Keywords: acceleration of particles, turbulence, plasma, cosmic rays, supernova remnants.

1 Introduction

Turbulence have long been regarded as particle accelerator. There are mainly two acceleration mechanisms by turbulence. One is due to wave-particle interactions, where the particle mean free path is comparable to or larger than the wavelength of electromagnetic fluctuations [1, 2]. The other is due to large-scale fluctuations of plasma flows, where the particle mean free path is smaller than turbulent scales [3]. Furthermore, turbulence is generally divided by compressible and incompressible modes. Therefore, particle accelerations by turbulence (the so called second order acceleration) are classified into four types:

- acceleration by small-scale compressible modes
- acceleration by large-scale compressible modes
- acceleration by small-scale incompressible modes
- acceleration by large-scale incompressible modes

First three acceleration mechanisms have already investigated in many paper. The last acceleration mechanism, acceleration by large-scale incompressible turbulence (turbulent shear acceleration), has not been investigated in detail, while Bykov & Toptygin (1983) have briefly discussed the turbulent shear acceleration [4]. For the acceleration by small-scale incompressible turbulence, that is, the second order acceleration by Alfvén waves, particles are accelerated by Alfvén waves propagating with the Alfvén velocity, v_A . The velocity fluctuation by Alfvén mode, δu , is represented by $\delta u = v_A(\delta B/B_0)$, where δB and B_0 are the fluctuated and mean magnetic fields, respectively. For the super Alfvénic turbulence, δB becomes larger than B_0 and the velocity fluctuations, δu , become larger than the Alfvén velocity, v_A . Then, particles could be more efficiently accelerated by the magnetic field fluctuations propagating (advecting) with a plasma velocity larger than the Alfvén velocity. Therefore, the turbulent shear acceleration could be more efficient than the second order acceleration by the Alfvén waves.

Particle acceleration by a simple shear flow has already investigated [5, 6]. However, shear flows are potentially unstable to the Kelvin-Helmholtz instability and produce

turbulence. Therefore, the turbulent shear acceleration is expected to be important. In this paper, we investigate the turbulent shear acceleration by considering ensemble average of an extended transport equation which includes particle acceleration by shear flows.

2 Ensemble-Averaged Transport Equation

In this section, we briefly derive the ensemble-averaged transport equation of energetic particles. The detailed derivation can be found in [7]. Propagation and acceleration of energetic charged particles in a plasma flow are described by a transport equation. Parker (1965) derived the transport equation which includes spatial diffusion, convection, and adiabatic acceleration [8]. After that his work was extended by several authors. For isotropic diffusion and a nonrelativistic plasma flow, the extended transport equation is given by (Equation (4.5) of [9])

$$\begin{aligned} \frac{\partial F}{\partial t} &+ U_i \frac{\partial F}{\partial x_i} - \frac{\partial}{\partial x_i} \left(\kappa \frac{\partial F}{\partial x_i} \right) \\ &- \frac{p}{3} \frac{\partial U_i}{\partial x_i} \frac{\partial F}{\partial p} - \frac{1}{p^2} \frac{\partial}{\partial p} \left(\kappa \Gamma \frac{p^4}{v^2} \frac{\partial F}{\partial p} \right) \\ &- \frac{1}{p^2} \frac{\partial}{\partial p} \left(\kappa \frac{DU_i}{Dt} \frac{DU_i}{Dt} \frac{p^4}{v^4} \frac{\partial F}{\partial p} \right) \\ &+ \frac{\partial}{\partial x_i} \left(\kappa \frac{DU_i}{Dt} \frac{p}{v^2} \frac{\partial F}{\partial p} \right) \\ &+ \frac{1}{p^2} \frac{\partial}{\partial p} \left(\kappa \frac{DU_i}{Dt} \frac{p^3}{v^2} \frac{\partial F}{\partial x_i} \right) = 0, \end{aligned} \quad (1)$$

where Γ is defined by

$$\Gamma = \frac{1}{5} \left(\frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} \right) - \frac{2}{15} \frac{\partial U_i}{\partial x_i} \frac{\partial U_j}{\partial x_j}. \quad (2)$$

$F(p, \mathbf{x}, t)$, x_i , U_i , $\kappa(p)$ are the distribution function, position, plasma velocity, and spatial diffusion coefficient, respectively. v and p are the particle velocity and four momentum in the fluid rest frame, respectively. The spatial diffusion coefficient, κ , is represented by $\kappa(p) = \tau(p)v^2/3$ for isotropic diffusion, where $\tau(p)$ is the mean scattering

time and τ_v is the particle mean free path. The first four terms of Equation (1) are the same as the Parker equation and the fifth term describes the shear acceleration.

In order to understand essential features of the turbulent shear acceleration, we neglect the spatial transport and we assume that the plasma velocity field, $U_i(\mathbf{x}, t)$, is static, random, statistically homogenous and isotropic incompressible turbulence, that is, $U_i = \delta u_i(\mathbf{x})$ and $\langle \delta u_i \rangle = 0$, where $\langle \dots \rangle$ denotes ensemble average. The correlation function of the plasma velocity field is given by

$$\langle \delta u_i(\mathbf{x}) \delta u_j(\mathbf{x}') \rangle = \int \frac{d^3k}{(2\pi)^3} K_{ij}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{x}_l - \mathbf{x}'_l)} , \quad (3)$$

and

$$K_{ij}(\mathbf{k}) = S(k) \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) , \quad (4)$$

where k and $S(k)$ are the wavenumber, frequency, and spectrum of incompressible turbulence, respectively.

The distribution function of particles can also be divided by an ensemble-averaged component and a fluctuated one, that is, $F = f + \delta f$ and $\langle F \rangle = f$. Then, the ensemble-averaged transport equation can be represented by

$$\frac{\partial f}{\partial t} - \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_{\text{TSA}} \frac{\partial f}{\partial p} \right) = 0 , \quad (5)$$

where we neglect the spatial transport and the momentum diffusion coefficient, $D_{\text{TSA}}(p)$, is given by

$$D_{\text{TSA}}(p) = \frac{2}{9} p^2 \tau(p) \int \frac{d^3k}{(2\pi)^3} S(k) k^2 \left(\frac{3}{5} + \frac{\langle \delta u^2 \rangle}{v^2} \right) . \quad (6)$$

We here consider turbulence with a large lengthscale compared with the particle mean free path, τ_v , so that the upper limit of k -integral should be limited by $\min\{k_{\text{max}}, k_{\text{res}}\}$ where k_{max} is the maximum wavenumber of turbulence and $k_{\text{res}} \approx (\tau_v)^{-1}$. The momentum diffusion coefficient, D_{TSA} , is dominated by small scale turbulence when $k^5 S(k)$ is an increasing function of k . Therefore, the turbulent shear acceleration becomes important in the small scale for a Kolmogorov-like spectrum, $S(k) \propto k^{-11/3}$.

3 Analytical solution

In this section, we present specific expressions of the momentum diffusion coefficient and analytical solutions of the ensemble-averaged transport equation for simple velocity spectra. We especially focus on the turbulent shear acceleration of relativistic particles ($v \approx c$) in nonrelativistic turbulence ($\langle \delta u^2 \rangle \ll c^2$), so that we neglect the terms of v^{-2} in Equation (6). We here assume a functional form of the mean scattering time, $\tau(p)$, to be $\tau_0(p/p_0)^\alpha$, where p_0 and τ_0 are the initial four momentum and the mean scattering time of particles with p_0 , respectively. To make the expression simple, hereafter the four momentum, time, and momentum diffusion coefficient are normalized by p_0 , τ_0 , and $p_0^2 \tau_0^{-1}$, respectively. Normalized quantities are denoted with a tilde.

For a static monochromatic spectrum of incompressible turbulence, $S(k)$ is given by

$$S(k) = \frac{\langle \delta u^2 \rangle (2\pi)^2}{4k_0^2} \delta(k - k_0) . \quad (7)$$

From Equations (6) and (7), the momentum diffusion coefficient is represented by

$$\tilde{D}_{\text{TSA}} = \frac{(\tau_0 v k_0)^2 \langle \delta u^2 \rangle}{15 v^2} \tilde{p}^{2+\alpha} . \quad (8)$$

For a static Kolmogorov-like spectrum of incompressible turbulence, we assume that $S(k)$ is given by

$$S(k) = \frac{\langle \delta u^2 \rangle (2\pi)^2}{6 \left(k_0^{-2/3} - k_{\text{max}}^{-2/3} \right)} k^{-11/3} \quad (\text{for } k_0 \leq k \leq k_{\text{max}}) \quad (9)$$

Then, from Equations (6) and (9), the momentum diffusion coefficient is represented by

$$\tilde{D}_{\text{TSA}} \approx \frac{(\tau_0 v k_0)^2 \langle \delta u^2 \rangle}{30 v^2} \tilde{p}^{2+\alpha} \left(\frac{\min\{k_{\text{max}}, k_{\text{res}}\}}{k_0} \right)^{4/3} , \quad (10)$$

where we have assumed $k_0 \ll \min\{k_{\text{max}}, k_{\text{res}}\}$. The factor, $(\min\{k_{\text{max}}, k_{\text{res}}\}/k_0)^{4/3}$, is expected to be large. Therefore, the Kolmogorov-like turbulent cascade enhances the turbulent shear acceleration. For $k_{\text{res}} < k_{\text{max}}$, \tilde{D}_{TSA} is represented by

$$\tilde{D}_{\text{TSA}} \approx \frac{(\tau_0 v k_0)^{2/3} \langle \delta u^2 \rangle}{30 v^2} \tilde{p}^{2-\alpha/3} . \quad (11)$$

Therefore, the momentum diffusion coefficient can be represented by $\tilde{D}_{\text{TSA}} = \tilde{D}_0 \tilde{p}^{2+\beta}$ for above simple cases, where $\beta = \alpha$ for the monochromatic spectrum and the Kolmogorov spectrum of the case $k_{\text{res}} > k_{\text{max}}$, and $\beta = -\alpha/3$ for the Kolmogorov spectrum of the case $k_{\text{res}} < k_{\text{max}}$.

We next discuss analytical solutions of the ensemble-averaged transport equation. We assume that particles are uniformly distributed in the three dimensional space and injected at time, $\tilde{t} = 0$, with the four momentum, $\tilde{p} = 1$. Then, the ensemble-averaged transport equation is represented by

$$\frac{\partial f}{\partial \tilde{t}} - \frac{1}{\tilde{p}^2} \frac{\partial}{\partial \tilde{p}} \left(\tilde{p}^2 \tilde{D}_{\text{TSA}} \frac{\partial f}{\partial \tilde{p}} \right) = \frac{N}{4\pi} \delta(\tilde{t}) \delta(\tilde{p} - 1) , \quad (12)$$

where N is the number of injected particles. If the momentum diffusion coefficient is represented by $\tilde{D}_{\text{TSA}} = \tilde{D}_0 \tilde{p}^{2+\beta}$, for $\beta \neq 0$, the solution is given by [10, 11]

$$f(\tilde{p}, \tilde{t}) = \frac{N}{4\pi |\beta| \tilde{D}_0 \tilde{t}} \tilde{p}^{-(3+\beta)/2} \exp \left(-\frac{1 + \tilde{p}^\beta}{\beta^2 \tilde{D}_0 \tilde{t}} \right) \times I_{|1+3/\beta|} \left[\frac{\tilde{p}^{-\beta/2}}{\beta^2 \tilde{D}_0 \tilde{t}} \right] , \quad (13)$$

where I_ν is the modified Bessel function of the first kind. The solution approaches $\tilde{p}^3 f \propto \tilde{p}^{-\beta}$ for $\tilde{p} > 1$. For $\beta = 0$, the solution is given by [11]

$$f(\tilde{p}, \tilde{t}) = \frac{N}{(4\pi)^{3/2} \sqrt{\tilde{D}_0 \tilde{t}}} \exp \left\{ -\frac{(\ln \tilde{p} + 3\tilde{D}_0 \tilde{t})^2}{4\tilde{D}_0 \tilde{t}} \right\} , \quad (14)$$

and the evolution of the mean momentum, $\tilde{p}_m(\tilde{t}) = N^{-1} \int \tilde{p} f(\tilde{p}, \tilde{t}) 4\pi \tilde{p}^2 d\tilde{p}$, is given by

$$\tilde{p}_m(\tilde{t}) = \exp(4\tilde{D}_0 \tilde{t}) . \quad (15)$$

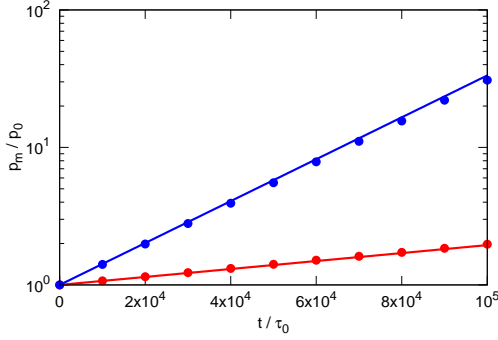


Fig. 1: Time evolution of the mean four momentum for $\alpha = 0$ and $\tau_0 ck_0 = 10^{-1}$. The dots and solid lines show the results of Monte Carlo simulations and analytical solutions (Equations (8), (10) and (15)), respectively. The red and blue show cases of the monochromatic and Kolmogorov spectra with $\tau_0 ck_{\max} = 10^{-1/3}$, respectively.

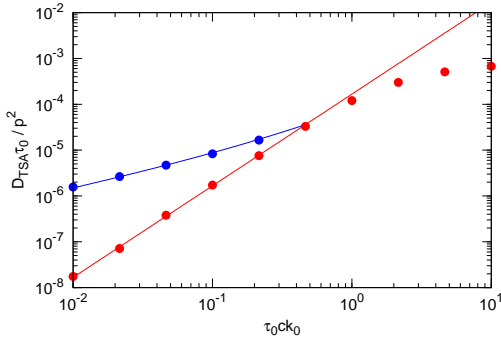


Fig. 2: Wavenumber dependence of the momentum diffusion coefficient for $\alpha = 0$. The dots and solid lines show the results of Monte Carlo simulations and analytical solutions of Equations (8) and (10), respectively. The red and blue show cases of the monochromatic and Kolmogorov spectra with $\tau_0 ck_{\max} = 10^{-1/3}$, respectively.

Note that solutions of Equations (13) and (14) are not valid for $\tilde{t} \ll 1$ and $\tilde{p} \gg 1$ because of causality.

In [7], using Monte Carlo simulations, we have confirmed that the ensemble-averaged transport equation describes the turbulent shear acceleration. Figure 1 shows the evolution of the mean four momentum for $\alpha = 0$ and $\tau_0 ck_0 = 10^{-1}$. Monte Carlo simulation results are in good agreement with the analytical solutions of Equations (8), (10) and (15). By comparing the growth rate of the mean momentum of simulation particles with Equation (15), we can obtain the momentum diffusion coefficient of Monte Carlo simulations. Figure 2 shows the wavenumber dependence of the momentum diffusion coefficient, $\tilde{D}_0 = D_{\text{TSA}} \tau_0 / p^2$ for $\alpha = 0$. Simulation results are in good agreement with Equations (8) and (10) as long as $\tau_0 ck_0 < 1$, but simulation results for the monochromatic spectrum deviate from Equation (8) at $\tau_0 ck_0 > 1$. As already mentioned in Section 2, this is because our treatment is not valid when the particle mean free path is larger than the turbulent scale. Furthermore, we have confirmed that the Kolmogorov-like turbulent cascade (blue) enhances the turbulent shear acceleration.

4 Discussion

We first discuss another important effect of turbulence on the particle transport. Bykov & Toptygin (1993) show that turbulence enhances spatial diffusion [3]. For strong turbulence, κ_{turb} becomes of the order of $L_0 \sqrt{\langle \delta u^2 \rangle}$ [3], where L_0 is the injection lengthscale of turbulence, so that spatial diffusion of particles with a small mean free path is dominated by turbulent diffusion and an energy-independent diffusion is realized. The ratio of the turbulent diffusion and the Bohm diffusion, κ_{Bohm} , is given by

$$\frac{\kappa_{\text{turb}}}{\kappa_{\text{Bohm}}} \sim 3 \times 10^6 \left(\frac{p}{m_p c} \right)^{-1} \left(\frac{\sqrt{\langle \delta u^2 \rangle}}{c} \right) \times \left(\frac{B}{1 \mu\text{G}} \right) \left(\frac{L_0}{1 \text{pc}} \right), \quad (16)$$

where m_p and B are the proton mass and magnetic field, respectively. Therefore, turbulent diffusion of energetic particles could be important in SNRs, PWNe, astrophysical jets, etc.

From Equation (11), the acceleration timescale, $t_{\text{acc}} = p^2 / D_{\text{TSA}}$, of the turbulent shear acceleration for the Kolmogorov spectrum of the case $k_{\text{res}} < k_{\text{max}}$ is represented by

$$\begin{aligned} t_{\text{acc}} &= \frac{30}{\{\tau(p) v k_0\}^{2/3}} \frac{v^2}{\langle \delta u^2 \rangle} \tau(p) \\ &= 9 \times 10^6 \text{ sec} \\ &\quad \times \left(\frac{p}{m_p c} \right)^{1/3} \left(\frac{\langle \delta u^2 \rangle}{c^2} \right)^{-1} \\ &\quad \times \left(\frac{B}{1 \mu\text{G}} \right)^{-1/3} \left(\frac{L_0}{1 \text{pc}} \right)^{2/3}, \quad (17) \end{aligned}$$

where we have assumed $L_0 = 2\pi/k_0$ and the Bohm diffusion, $\tau(p) = p/(eB)$, in the last equation. Therefore, particles can be accelerated to relativistic energies by large-scale turbulence in many astrophysical objects. In addition, if particles are initially accelerated at the shock, large-scale turbulence can change energy spectra of the accelerated particles in the shock downstream region.

Next, we compare the turbulent shear acceleration and the second order acceleration by Alfvén waves. The momentum diffusion coefficient of the second order acceleration by Alfvén waves is given by $D_A \sim p^2 v_A^2 / (9\kappa)$. The ratio of the turbulent shear acceleration and the second order acceleration by Alfvén waves is given by

$$\frac{D_{\text{TSA}}}{D_A} \sim \frac{\langle \delta u^2 \rangle}{v_A^2} \left(\frac{\tau v}{L_0} \right)^{2/3}, \quad (18)$$

where we have adopted Equation (11) as D_{TSA} . Therefore, the turbulent shear acceleration could be more efficient than the second order acceleration by Alfvén waves for super-Alfvénic turbulence ($\sqrt{\langle \delta u^2 \rangle} > v_A (\tau v / L_0)^{-1/3}$). In other words, the turbulent shear acceleration becomes important when there are strong magnetic field fluctuations ($\delta B / B_0 > 1$) because the plasma velocity fluctuation by Alfvén waves, δu , is represented by $\delta u = v_A (\delta B / B_0)$.

5 Summary

In this paper, we have derived a particle transport equation averaged over random plasma flows in order to understand

particle acceleration in incompressible turbulence with a larger lengthscale than the particle mean free path. We have considered ensemble average of the extended transport equation provided by [9]. This is a simple extension of previous work that considered ensemble average of the transport equation provided by [8]. We have found that the turbulent shear acceleration by incompressible turbulence becomes important in small scale for Kolmogorov-like turbulence. Recent simulations show that turbulence is produced in many astrophysical objects, so that turbulent diffusion and turbulent acceleration are expected to be important.

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References

- [1] J. Skilling, 1975, MNRAS, 172, 557
- [2] R. Schlickeiser and J. A. Miller, 1998, ApJ, 492, 352
- [3] A. M. Bykov and I. N. Toptygin, 1993, Phys.–Usp., 36, 1020
- [4] A. M. Bykov and I. N. Toptygin, 1983, Proc. 18th ICRC (India), 9, 313
- [5] E. G. Berezhko and G. F. Krymskii 1981, Soviet Astron. Lett., 7, 352
- [6] J. A. Earl, J. R. Jokipii, and G. Morfill, 1988, ApJ, 331, L91
- [7] Y. Ohira, 2013, ApJ, 767, L16
- [8] E. N. Parker, 1965, Planet. Space. Sci, 13, 9
- [9] G. M. Webb, 1989, ApJ, 340, 1112
- [10] E. G. Berezhko, 1982, Soviet Astron. Lett., 8, 403
- [11] F. M. Rieger and P. Duffy, 2006, ApJ, 652, 1044