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EARTH AS A LOW FREQUENCY GRAVITATIONAL WAVE DETECTOR

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Abstract

The effects of global and seismic perturbations of the Earth induced by a cosmic gravitational radiation is discussed. The history and brief review of the subject of interest is presented. The model of spherical homogeneous isotropic Earth is taken for to calculate perturbations of both type: mode oscillations and acoustic waves on the Earth's surface. The last one is reduced from general solution through the limit transition to infinite globe radius. Then a four component seismic array on the land is considered and optimal data processing algorithms for the filtering of short GW-signals are analyzed. Further a half-year seismic data record from the seismic net "TERRASCOPE" is utilized for to get upper limit estimations of GW-burst intensity integrated over the low-frequency range of $0.01 - 0.1 \text{ Hz}$. The best value of upper limit is $h_{+up} \simeq (1.0 \pm 0.1) \times 10^{-14}$ in terms of dimensionless metric perturbations. The equivalent conversion of $h_{+up}$ to the spectral intensity of gravitational radiation flux density $I_g$ gives the dominating value of upper limit $I_g < 1.6 \cdot 10^6 \text{ erg/(s \cdot cm}^2 \cdot \text{ Hz)}$. The correspondent amplitude of seismo-acoustic perturbation less $1.2 \cdot 10^{-8} \text{ cm at the frequency } f_g = 0.05 \text{ Hz}$, for example.
1. Introduction

The idea of seismic detection of gravitational wave (initially proposed by J. Weber) was analyzed in detail by Dyson (1968) for pulsating radio sources. The response of the Earth to a gravitational wave in 1-Hz band was calculated by Dyson, first for a stationary flat Earth and then for a spherical rotating Earth. According to Dyson the perturbation of the elastic medium to the plane gravitational wave \( A_\omega \) with complex amplitude \( a \), angular frequency \( \omega \) and wave-vector

\[
k^j = (\omega/c) [ \sin \theta, 0, \cos \theta ], \tag{1.1}
\]

coming at an angle \( \theta \) to the inward normal will be a superposition of two plane waves, one longitudinal \( y^j_L \) and one transverse \( y^j_T \), running away from the surface with velocities \( u \) and \( s \). Those elastic waves propagate almost perpendicularly to the surface.

The formula obtained by Dyson (1968)

\[
y^j = \frac{1}{2} i a (s/\omega) \sin \theta [ \cos \theta, i, -(s/u) \sin \theta ] \tag{1.2}
\]
gives complete information concerning the phase, amplitude and direction of seismic displacement induced at the Earth's surface by a gravitational wave. In the long-wave approximation \( (u, s \ll c) \) all points on the surface must move together in phase and all seismographs in a seismic array should respond coherently to an incident gravitational wave provided that the extent of the array is small compared with \( (2\pi c/\omega) \), where \( c \) is the speed of light.

According to Dyson (1968) the perturbations induced by gravitational radiation of pulsars are about \( 10^3 \) below natural seismic noise levels, but the uncertainties in this estimate are great enough thus an attempt to detect pulsars seismically is not absolutely hopeless.

On the other hand, the reliable sources of gravitational radiation (Thorne, 1987) are nearby double stars which emit waves with frequencies comparable to the fundamental frequency of spheroidal oscillations of the Earth.

For this reason it was interesting to calculate the elastic response of the Earth to monochromatic gravitational waves of frequencies in the range from \( \sim 10^{-4} \) Hz to 1 Hz. This calculations have been made by Ashby and Dreitlein (1975) initially for an uniform spherical detector having properties (mass, elastic modules) comparable to those of the real Earth and then a more complicated problem of self stressed sphere has been solved. It was obtained that gravitational wave must excite only spherical normal modes of the Earth \( S_{nm} \) types.

Recently the idea of seismic detection gravitational wave was revived, but for detection of GW-bursts from relativistic pulse sources (Braginskii et al., 1985; Rudenko, 1968) at low-frequency range of \( \sim 0.01 - 0.1 \) Hz. The astrophysical forecast for intensity of GW-bursts in this frequency
region is $h = 10^{-18} - 10^{-17}$ in term of metric's variations. This is much larger then $h = 10^{-21} - 10^{-22}$ for the typical bar detectors with frequency $\sim 1$ kHz and $h = 10^{-23} - 10^{-24}$ for the pulsating radio sources range with frequency $\sim 1$ Hz. An advantage of this project is also so called a "window of transparence" in seismic noise spectrum at frequency range of $\sim 0.01 - 0.1$ Hz.

A possibility of the low-frequency GW-detection through responses of seismic array placed on the surface of the Earth has been studied in the paper (Gusev et al., 1990; Gusev and Kravchuk, 1994; Kravchuk et al., 1995). It was proposed several algorithms for optimal filtration of the GW-signals on a seismic noise background which depressed correlated seismic noises for special models of the Earth's crust and seismic array structure. In this paper we present the results of application of these recommendations to the raw digital seismic data. The model of spherical homogeneous isotropic Earth is taken for to calculate perturbations of both type: mode oscillations and acoustic waves on the Earth's surface. The last one is reduced from general solution through the limit transition to infinite globe radius. The response of the Earth in the limit of high frequency (frequency high compared with the fundamental resonance frequencies of the Earth's sphere) or in the limit transition to infinite globe radius is found with preserving of dependence of longitudinal and transverse wave magnitudes upon spherical coordinates. Then a three and four component seismic array on the land is considered and optimal data processing algorithms for the filtering of short GW-signals are analysed. Further a half-year digital synchronous seismic data record from the seismic net "TERRASCOPE" (California, USA) is utilized for to get upper limit estimations of GW-burst intensity integrated over the low-frequency range of $\sim 0.01 - 0.1$ Hz.

So our main goal is to calculate an upper limit on an intensity gravitational wave (GW) bursts on the basis of raw digital seismic data, optimal data processing algorithms and model of the Earth mentioned above.

2. Brief review of the subject of interest

There are several papers (Wiggins and Press, 1969; Sadeh and Meidav, 1972; Vinnik, 1972; Mest et al., 1972; Kolesnikov et al., 1974) devoted to attempts to discover of cosmic origin in a seismic data at frequency range about 1 Hz.

For pulsars as the sources with very high stable frequency one needs only to accumulate data from existing seismometers over a long enough time, and to look for Fourier components at the pulsating radio source frequencies. This method of the analysing was used by all authors mentioned above.

The first an effort to detect seismic signals at pulsar frequencies by means of the large aperture seismic array (LASA) in Montana was made by Wiggins and Press (1969). LASA date were scanned over a 20.5 hour time interval on June 4, 1968. The spectral amplitudes of the sum
trace for LASA were determined at several pulsar frequencies and also at the double- and half-frequencies of CP0950, CP1133, CP0834, CP1919 (see the table 1). No correlation between pulsar frequencies and spectral peaks was found expect for CP1133. However, the presence of nearby spectral peaks with the same or larger amplitude did not permit to attach any significance to this result. The only positive statement which have been made is that, if ground motion was excited by gravitational waves, then the amplitudes was smaller than about $10^{-9}$ cm.

The paper by Mast et al. (1972) was also devoted to the search of periodical signal in microseismic noise on the double- frequency of CP1133. The authors utilized the data from vertical seismometer placed nearby Jamestown (California, USA) and the method of synchronous accumulating of the seismic signals. After the processing of $30333 \times 2$ seismic data authors had made the conclusion about an absence of any periodical signal at the double- frequency of CP1133. The upper estimate for the expected displacement was about $10^{-10}$ cm and the power spectrum of Earth noise at more quite time intervals was about of $1.0 \times 10^{-3}$/Hz$^{-1/2}$.

Table 1. Basic, double- and half- frequencies of pulsars analyzed by Wiggins and Press (1969), where $\Omega_p$ is the basic pulsar frequency.

<table>
<thead>
<tr>
<th>Pulsar</th>
<th>$\Omega_p$, Hz</th>
<th>$\Omega_p/2$, Hz</th>
<th>$2\Omega_p$, Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP0834</td>
<td>0.785075</td>
<td>0.392537</td>
<td>1.570149</td>
</tr>
<tr>
<td>CP1919</td>
<td>0.747775</td>
<td>0.373887</td>
<td>1.495550</td>
</tr>
<tr>
<td>CP0950</td>
<td>3.951552</td>
<td>1.975776</td>
<td>7.903103</td>
</tr>
<tr>
<td>CP1133</td>
<td>0.841815</td>
<td>0.420908</td>
<td>1.683630</td>
</tr>
</tbody>
</table>

The upper limit on the gravitational flux reaching the Earth from the GRB Pulsar NP0532 was estimated experimentally by Levin and Stebbins (1972). A 30-m laser strain meter located in an unworked gold mine near Boulder (Colorado) have been used for this aim.

The minimum detectable signal could be produced by an incident gravitational flux of $10^9$ ergs/($s \cdot cm^2$) and authors found no effect at this level. They concluded that if a pulsar-coherent strain signal is present, it's amplitude was less than $3 \times 10^{-17} (\Delta L/L)$. They also confirmed that this limit was as consistent with instrumental wide-band noise, that was about $3 \times 10^{-18} (\Delta L/L)^2$/Hz, with integration bandwidth of approximately $0.7 \times 10^{-3}$/Hz.

The table 2 summarizes all experiments mentioned above, where $T_{\text{int}}$ is the integration time interval, $f$ is the frequency under consideration, $\Delta y$ is the upper limit of seismo-acoustic perturbations induced by pulsar GW-signals, $h_{up}$ is the same upper limit on a gravitational radiation from pulsars but in terms of dimensionless metric's perturbations.
One of the first papers devoted to the coupling of gravitational waves to normal modes of the Earth is one of Forward et al. (1961). Analyzing the records of a Benioff's strain-seismograph in seismically quite time intervals, they have given the estimate of upper limit of gravitational radiation at the frequency of \( \sigma S_2 \) mode (period is about of \( \sim 54 \text{ min} \)):

\[
h_\nu < 2 \times 10^{-12} \text{ hour}^{1/2},
\]

where \( h_\nu \) is the gravitational wave spectral amplitude.

The spectral amplitudes of the Earth noises observed with LaCoste-Romberger gravimeter by Weber and Larson (1966) are in agreement with an estimate obtained by Forward et al. (1962).

Table 2. \textit{Brief history review of experiments devoted to attempts to discover harmonic signals induced by pulsars GW-radiation in seismic traces.}

<table>
<thead>
<tr>
<th>Author, year</th>
<th>Placement of device</th>
<th>Type of device</th>
<th>( T_{\text{int.}} ), s</th>
<th>( f_r ), Hz</th>
<th>( \Delta y ), cm</th>
<th>( h_\nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wiggins, Press, 1969</td>
<td>Montana, USA</td>
<td>LASA</td>
<td>73800</td>
<td>1.68</td>
<td>( \sim 10^{-7} )</td>
<td>( 5 \times 10^{-12} )</td>
</tr>
<tr>
<td>Shadch, Meidav, 1972</td>
<td>Ailat, Israel</td>
<td>vertical seismic meter</td>
<td>13107</td>
<td>1.69</td>
<td>( 2 \times 10^{-9} )</td>
<td>( \sim 10^{-15} )</td>
</tr>
<tr>
<td>Vinnik, 1972</td>
<td>Ust Kamenogorsk, Kazakhstan</td>
<td>idem</td>
<td>1260</td>
<td>1.65</td>
<td>( \sim 10^{-8} )</td>
<td>( 5 \times 10^{-13} )</td>
</tr>
<tr>
<td>Mast, Nelson, 1972</td>
<td>Jamestown, California, USA</td>
<td>idem</td>
<td>50666</td>
<td>1.68</td>
<td>( \sim 10^{-10} )</td>
<td>( 5 \times 10^{-15} )</td>
</tr>
<tr>
<td>Colesnikov et al., 1974</td>
<td>&quot;Naryn&quot;, USSR</td>
<td>idem</td>
<td>21000</td>
<td>1.69</td>
<td>( \sim 10^{-8} )</td>
<td>( 5 \times 10^{-14} )</td>
</tr>
<tr>
<td>Levin, Stebbins, 1972</td>
<td>Boulder, Colorado</td>
<td>laser strain meter</td>
<td>( 1.5 \times 10^8 )</td>
<td>30.2</td>
<td>( 4 \times 10^{-14} )</td>
<td>( 3 \times 10^{-17} )</td>
</tr>
</tbody>
</table>

There are more late works (Weber 1967; Burke 1973; Zimmerman and Hellings 1980). However the use of highly simplified models in addition to improper treatment of the coupling to high-frequency modes had led one to suspect these limits.

More correctly the background level of gravitational radiation consistent with the most complete model of the Earth and seismic data available in the literature was calculated by Boughn
and Kuhn (1984). They have used the six-shell model with no bulk dissipation (shear only) described by Masters and Gilbert (1983) and the power density values of vertical seismic oscillations described by Agnew and Berger (1978).

The limits on a stochastic gravitational wave background calculated by Boughn and Kuhn (1984) from observations of terrestrial oscillations are summarized in the table 3, where \( n \) is the Earth's mode number, \( \omega_{n2} \) and \( T_{n2} \) are correspondent angular frequencies and periods of normal mode oscillations of the Earth, \( \tau_{n2} \) and \( S_{n2} \) are correspondent damping times and overlap coefficients. These limits are from one to three orders of magnitude higher than that of Zimmerman and Hellings (1980) and Weber (1967) due to using of more realistic model of the Earth. However to authors opinion these "Earth's mode" limits were got due to the finite spectral resolution of seismic data (Agnew and Berger, 1978) and its might be decreased by two orders of magnitude or more if one would observe for long periods of time at a seismically quiet location.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \omega_{n2} ), s(^{-1} )</th>
<th>( T_{n2} ), min.</th>
<th>( \tau_{n2} ), s</th>
<th>( S_{n2} )</th>
<th>Flux, erg / ((\text{s} \cdot \text{cm}^2 \cdot \text{Hz}))</th>
</tr>
</thead>
</table>
| 0...
2...
4...
22...
28...
60...
| 0.00194
0.00899
0.01080
0.05160
0.06350
0.13480
| 53.9
17.5
9.7
2.0
1.6
0.78
| 6.8 \times 10^8
1.0 \times 10^8
1.0 \times 10^4
6.1 \times 10^4
1.2 \times 10^4
1.8 \times 10^4
| 7.0 \times 10^{-1}
9.7 \times 10^{-4}
8.7 \times 10^{-2}
1.7 \times 10^{-3}
4.8 \times 10^{-3}
-3.7 \times 10^{-3}
| 7.1 \times 10^9
1.6 \times 10^{15}
5.6 \times 10^7
1.7 \times 10^8
4.4 \times 10^{10}
3.4 \times 10^6 |

3. A seismic data records and preliminary data processing

A seismic data that have been used to determine once more the upper limit of gravitational wave background were digital synchronous data recorded by horizontal and vertical long-period seismographs placed at six seismic stations included to the "TERRASCOPE" seismic net (California, USA). High resolution vertical (LHZ) and horizontal (LHN and LHE) measuring channels of seismic meters are utilized over a period of time from August 01, 1993 to February 24, 1994. A sampling time was one second for all used seismographs. The main parameters of seismic stations (such as their names, codes, coordinates and altitude) of six seismic stations used to compose different seismic arrays are shown in the table 4.
The seismic data was recorded in binary seismic analysis code (SAC). The SAC is an American standard of seismic data records. Each daily seismic data record consists of $N = 86400$ samples. The transform function $W(p)$ of seismic ometers is expressed as

$$W(p) = A \cdot H(p)$$  \hspace{1cm} (3.3)

where $p$ is a Laplacian, and $A$ is the same constant having a dimension [count/(mm/s)], i.e. the velocity channels of seismographs is utilized. The form of $H(p)$ is determined by the set of its poles and zeroes.

For example the transform function of seismic meters placed at seismic stations with codes of GSC, ISA and PAS have got two zeroes and four poles. But ones placed at seismic stations with codes of RPV and USC have got two zeroes and five poles. For each channel of all used seismic meters there were known numerical values of constants $A$ and zeroes and poles. Thus we could easily calculate all amplitude-frequency characteristics of used seismic meters.

The preliminary seismic data processing have been used to prepare the digital seismic data in form suitable for an analysis. This procedure consists in the drift removing from seismic data and inverse band limited filtering in frequency band of $\sim 0.01 - 0.1$ Hz by using digital non-recursive filters. It has been used to obtain horizontal and vertical seismic vibrations of the ground, to improve a signal-noise ratio and to eliminate the influence of gain-frequency non-uniformity of the different seismographs. A Kaiser window weight function (Antoniou, 1979) has been used to obtain a good attenuation and a small flatness of the frequency response. The computer realization of a filtering process has been carried out by the overlapping segment with a summarizing method (Rabiner and Gold, 1975). The Fast Fourier Transform (FFT) has been used to accelerate substantially the process of calculation. Besides the process of the filtration included a special scaling of seismic data. The long periodical drift has been removed from the seismic data by least-square analysis (Korn, 1968).

Table 4. Parameters of six seismic stations included to the "TERRASCOPE" seismic net (California, USA).
<table>
<thead>
<tr>
<th>Name of seismic station</th>
<th>Code</th>
<th>Latitude, degree</th>
<th>Longitude, degree</th>
<th>Altitude, meter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goldston</td>
<td>GSC</td>
<td>35.303</td>
<td>-116.308</td>
<td>999</td>
</tr>
<tr>
<td>Isabella</td>
<td>ISA</td>
<td>35.663</td>
<td>-118.473</td>
<td>835</td>
</tr>
<tr>
<td>Pasadena</td>
<td>PAS</td>
<td>34.148</td>
<td>-118.172</td>
<td>295</td>
</tr>
<tr>
<td>Rancho Palos Verge</td>
<td>RPV</td>
<td>33.744</td>
<td>-118.404</td>
<td>0</td>
</tr>
<tr>
<td>Santa Barbara</td>
<td>SBC</td>
<td>34.442</td>
<td>-119.713</td>
<td>90</td>
</tr>
<tr>
<td>USC, Los Angeles</td>
<td>USC</td>
<td>34.021</td>
<td>-118.287</td>
<td>60</td>
</tr>
</tbody>
</table>

4. Response of the Earth to gravitational waves: the high frequency limit

In a normal Fermi coordinate system the basic response of a system of particles (each labeled by the index $p$) to a gravitational wave is given by the equations of geodesic derivation (Ashby and Dreitlein, 1975)

$$m_p \frac{d^2 x^m_p}{dt^2} = -m_p c^2 R^{m}_{\text{geo}} \omega^2 \frac{x^m_p}{R^m_p}, \ m = 1, 2, 3. \quad (4.4)$$

Such an equation indicates that the presence of the gravitational wave can be taken into account by adding to the Newtonian equations of motion a gravitational tidal force term, viz. $-m_p c^2 R^{m}_{\text{geo}} \omega^2 \frac{x^m_p}{R^m_p}$. In principal, the equations of motion of the elastic body of spherical symmetry can be derived from the equation of geodesic derivation mentioned above. But there are two another approaches of deriving the equations motion of the Earth under the influence of a gravitational wave. Both lead to the equation of motion which perhaps is most easily surmised from the equation (4).

In the first approach, the action principle is used to derive the equations of motion. The second method of deriving the equations of motion follows by using the exact equations

$$T^\mu_{\nu \mu} = 0, \quad (4.5)$$

which describe the movement of energy and momentum. The equation in more explicit form is

$$\frac{\partial T^{\mu \nu}}{\partial x^\rho} = -\Gamma^\mu_{\rho \nu} T^{\mu \nu} - \Gamma^\mu_{\mu \nu} T^{\rho \nu}. \quad (4.6)$$
The usual equations of motion are obtained from the momentum density equation:

\[ \frac{\partial T^{0m}}{\partial x^0} + \frac{\partial T^{mn}}{\partial x^m} = -\Gamma^m_{\mu \nu} T^{\mu \nu} - \Gamma^m_{\mu \nu} T^{\mu n}. \]  

(4.7)

In Newtonian approximation, the \( T^{0m} \) and \( T^{mn} \) on the left-hand side of the equation (7) are the nonrelativistic momentum density and the negative of the stress tensor, respectively.

To the same approximation, only \( T^{00} \) terms, which is equal to \( \rho c^2 \), need be retained on the right-hand side of the equation. The conventional definition of the elastic stress tensor is \( T^{mn} = -\sigma^{mn} \) and thus

\[ \frac{\partial T^{00}}{\partial x^0} = \frac{\partial \sigma^{00}}{\partial x^0} - \Gamma^0_{\nu 0} \rho c^2. \]  

(4.8)

The term \( \Gamma^0_{\nu 0} \) becomes in the long-wavelength approximation:

\[ \Gamma^0_{\nu 0} = -R_{\nu 0 0 \nu} \equiv -\frac{1}{2} h_{\nu \alpha 00} x^\alpha, \]  

(4.9)

where \( R_{\nu 0 0 \nu} \) is this Riemann tensor which a gravitational antenna detects.

Therefore, the equations of motion of the elastic continuum are

\[ \frac{\partial T^{0m}}{\partial x^0} = \frac{\partial \sigma^{mn}}{\partial x^m} + \frac{1}{2} \rho c^2 h_{m\nu 00} x^\nu, \]  

(4.10)

where the quantities \( T^{0m} \) (momentum density) and \( \sigma^{mn} \) (stress tensor) are ones familiar from the theory of elasticity.

If it is assumed that the excursions of the elements of the elastic body from equilibrium positions are small, the linear theory of elasticity is appropriate. Let the equilibrium positions of the elements of the elastic continuum be labeled by the coordinates \( x_i \). The displacement from equilibrium of an element at \( x_i \) is prescribed by the time-dependent vector \( u_m(x_i, t) \) which is assumed small. The momentum density is formed as the product \( \delta(\rho u_m)/\delta t \) and \( \rho \) can be considered to be a constant in a homogeneous medium.

The stress tensor \( \sigma^{mn} \) is related to the strain tensor (Landau and Lifshits, 1970) by a generalized Hooke's law, again under the assumption of small displacements. The strain of the elastic medium is defined by

\[ u_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]  

(4.11)
in Cartesian coordinates and is related to the stress by

\[ \sigma_{mn} = \delta_{ij} \lambda u_i + 2 \mu u_j, \]  

(4.12)

where \( \lambda \) and \( \mu \) are the Lame coefficients specifying the elastic properties of an isotropic medium.

The equations of motion of the elastic medium in linearized version are thus

\[ \rho \frac{\partial^2 u_m}{\partial t^2} = \frac{\partial}{\partial x_m} (\delta_{mn} \lambda u_n + 2 \mu u_m) + \frac{1}{2} \rho c^2 h_{mn,00} x^m. \]  

(4.13)

The boundary condition is that the total force per unit area at the surface of the elastic medium vanishes in the direction normal to the surface of the elastic body. If the normal vector is \( N = \{N^k\} \) then

\[ N^k \sigma_{kn} = 0 \]  

(4.14)

must be satisfied everywhere on the surface.

While the equations of motion have the gravitational driving force as a volume force a simple change of field variables leads to the effective consideration of the gravitational force as a surface force. Let us introduce the new variable \( Z_m \) by defining

\[ Z_m = u_m - \frac{1}{2} h_{mn} x^n. \]  

(4.15)

Ashby and Dreitzen (1975) have shown early that the new field variable \( Z_m \) can be directly measured by means of seismic meters in the absence of self-stress of the Earth. Since \( h_{mn} \) depends only upon time when evaluated at the center of mass of the elastic body, it follows that for the long-wavelength approximation

\[ Z_{mn} = u_{mn} - \frac{1}{2} h_{mn}. \]  

(4.16)

Consequently, the equations of motion assume the form

\[ \rho \frac{\partial^2 Z_m}{\partial t^2} = \frac{\partial}{\partial x_m} \left( \delta_{mn} \lambda Z_n + 2 \mu (Z_{mn} + \frac{1}{2} h_{mn}) \right). \]  

(4.17)

and since spatial derivatives of \( h_{mn} \) are neglected

\[ \rho \frac{\partial^2 Z_m}{\partial t^2} = \frac{\partial}{\partial x_m} \left( \delta_{mn} \lambda Z_n + 2 \mu Z_{mn} \right). \]  

(4.18)
The driving force has disappeared from the equations of motion only to appear in the boundary conditions

\[ N_{mn} \left( \delta_{mn} \lambda \ Z_{tt} + 2\mu \left( Z_{mn} + \frac{1}{2} \ h_{mn} \right) \right) = 0. \tag{4.19} \]

The equations of motion (18) are easiest to survey (Landau and Lifshitz, 1970) in a vector form:

\[ \rho \ \frac{\partial^2 Z}{\partial t^2} = (\lambda + 2\mu) \ \text{grad div} \ Z - \mu \ \text{rot rot} \ Z. \tag{4.20} \]

If the longitudinal and transverse vector fields are introduced by the decomposition

\[
\begin{align*}
Z &= Z^{(l)} + Z^{(t)}, \\
\text{div} \ Z^{(l)} &= 0, \quad \text{rot} \ Z^{(l)} = 0,
\end{align*}
\tag{4.21}
\]

then on the basis of Helmholtz's theorem the equation of motion breaks up into two wave equations:

\[
\begin{align*}
\frac{\partial^2 Z^{(l)}}{\partial t^2} &= c_l^2 \ \nabla^2 Z^{(l)}, \quad c_l = (\mu/\rho)^{1/2}, \\
\frac{\partial^2 Z^{(t)}}{\partial t^2} &= c_t^2 \ \nabla^2 Z^{(t)}, \quad c_t = ((\lambda + 2\mu)/\rho)^{1/2}.
\end{align*}
\tag{4.22}
\]

If the plane gravitational \( h_{mn} \) (in rectangular Cartesian geocentric coordinate system XYZ) propagates along the \( Z \)-direction then in spherical coordinates \((r, \theta, \varphi)\) the correspondent tensor components are

\[
\begin{align*}
h_{rr} &= \sin^2 \theta \ (h_{xx} \cos 2\varphi + h_{xy} \sin 2\varphi), \\
h_{r\theta} &= \sin \theta \ \cos \theta \ (h_{xx} \cos 2\varphi + h_{xy} \sin 2\varphi) \\
&= \frac{1}{2} \ \frac{\partial^2 h_{rr}}{\partial \theta^2}, \\
h_{r\varphi} &= \sin \theta \ (-h_{xx} \sin 2\varphi + h_{xy} \cos 2\varphi) \\
&= \frac{1}{2 \ \sin \theta} \ \frac{\partial h_{rr}}{\partial \varphi}.
\end{align*}
\tag{4.23}
\]

The stress components which occur in the boundary conditions in spherical coordinates assume next form

\[
\begin{align*}
\sigma_{rr} &= 2\mu \ Z_{rr} + \lambda \ \text{div} \ Z, \\
\sigma_{r\theta} &= 2\mu \ Z_{r\theta}, \\
\sigma_{r\varphi} &= 2\mu \ Z_{r\varphi},
\end{align*}
\tag{4.24}
\]
where

\[ Z_{rr} = \frac{\partial Z_r}{\partial r}, \]
\[ Z_{r\theta} = \frac{1}{2} \left( \frac{\partial Z_\theta}{\partial r} - \frac{Z_\theta}{r} + \frac{1}{r} \frac{\partial Z_r}{\partial \theta} \right), \]
\[ Z_{r\varphi} = \frac{1}{2} \left( \frac{1}{\sin \theta} \frac{\partial Z_r}{\partial \varphi} + \frac{\partial Z_\varphi}{\partial r} - \frac{Z_\varphi}{r} \right). \] (4.25)

The equations which must be satisfied on the surface of the sphere where \( r = R \) are

\[ 2\mu \left( Z_{rr} + \frac{1}{2} h_{rr} \right) + \lambda \div Z = 0, \]
\[ 2\mu \left( Z_{r\theta} + \frac{1}{2} h_{r\theta} \right) = 0, \]
\[ 2\mu \left( Z_{r\varphi} + \frac{1}{2} h_{r\varphi} \right) = 0. \] (4.26)

It was shown by Ashby and Dreitlein (1975) that the desired solution of (20) must be a linear combination of two vector fields:

\[ Z = C_1 \cdot Z^{(1)} + C_2 \cdot Z^{(2)}. \] (4.27)

The longitudinal vector field is explicitly

\[ Z^{(1)} = \frac{\exp \{ i \omega t \}}{q^2} \left\{ \hat{t} \frac{d J_2(qr)}{dr} S_2 + \hat{\theta} \frac{J_2(qr)}{r} \frac{\partial S_2}{\partial \theta} + \hat{\varphi} \frac{J_2(qr)}{r} \frac{1}{\sin \theta} \frac{\partial S_2}{\partial \varphi} \right\}, \] (4.28)

while the exited transverse vector field is

\[ Z^{(2)} = \frac{\exp \{ i \omega t \}}{k^2} \left\{ \hat{r} \frac{6 J_2(kr)}{r} S_2 + \frac{d(r J_2(kr))}{dr} \frac{1}{r} \left( \hat{\theta} \frac{\partial S_2}{\partial \theta} + \hat{\varphi} \frac{1}{\sin \theta} \frac{\partial S_2}{\partial \varphi} \right) \right\}. \] (4.29)

Here \( k = \omega/c \) and \( q = \omega/c \) are the wave numbers, \( \omega \) is the frequency of the gravitational wave, \( J_2(qr) \) and \( J_2(kr) \) are spherical Bessel functions. The auxiliary function \( S_2(\theta, \varphi) \) is given by

\[ S_2 = \frac{1}{2} \sin^2 \theta (a \cos 2\varphi + b \sin 2\varphi), \] (4.30)

where \( a \) and \( b \) can be chosen for waves of arbitrary polarization as complex quantities with the restriction

\[ |a|^2 + |b|^2 = 1. \] (4.31)
The contribution of waves with polarisation of \( h_+ \) and \( h_\times \) into intensity of incident GW is determined by values of \( a \) and \( b \).

The two coefficients \( C_1 \) and \( C_2 \) which obey the boundary conditions (26) are now completely determined and specify uniquely the response of the elastic sphere to a gravitational wave. One need only solve the two simultaneous linear algebraic equations

\[
\begin{align*}
C_1 \cdot (12\mu f_1(kR)) + C_2 \cdot (2\mu f_3(qR) - \lambda J_3(qR)) &= -2\mu h, \\
C_1 \cdot \left( \frac{1}{2} f_3(kR) + 2 f_0(kR) \right) + C_2 \cdot (f_3(qR)) &= -\frac{1}{2} h,
\end{align*}
\]  

(4.32)

where the auxiliary functions \( f_0, f_1, f_3 \) are

\[
\begin{align*}
f_0(x) &= J_0(x)/x^2, \\
f_1(x) &= \frac{d}{dx} \left( \frac{J_0(x)}{x} \right), \\
f_3(x) &= \frac{d^2}{dx^2} \left( \frac{J_0(x)}{x} \right).
\end{align*}
\]  

(4.33)

The expressions for longitudinal (28) and transverse (29) vector fields and simultaneous equations (32) were first written by Ashby and Dreitlein (1975). But the coefficients \( C_1 \) and \( C_2 \) cannot be found because the determinant of (32) vanishes at a natural resonance frequencies of the elastic system. For such a situation, it becomes important to treat the damping of the free vibrations of the sphere. Though, we take an interest in the solve of (32) only in the limit of high frequency (frequency high compared with the fundamental resonance frequencies of sphere) or in the limit transition to infinite globe radius, i.e. when next restrictions for wave vectors \( q \) and \( k \)

\[
|qR| \gg 1, \quad |kR| \gg 1
\]  

(4.34)

are valid. In this limit the approximate solution of (32) must ever exist.

Really, if we use an asymptotic behaviour of spherical Bessel function \( J_2(x) \) at \( |z| \gg 1 \) (Korn, 1968):

\[
J_2(x) \approx \frac{1}{x} \cos \left( x - \frac{2}{3} x \right) \approx -\frac{\sin x}{x},
\]  

(4.35)

one can find, keeping only leading terms in order of \( 1/\omega \), that there are valid the next approaches

\[
\begin{align*}
f_0(x) &\approx -\frac{\sin x}{x^3} \approx 0, \\
f_1(x) &\approx -\frac{\cos x}{x^2} \approx 0, \\
f_3(x) &\approx \frac{\sin x}{x} \approx -J_2(x), \\
\frac{d}{dx} \left( J_2(x) \right) &\approx -\frac{\cos x}{x}, \\
\frac{d}{dx} \left( x \cdot J_2(x) \right) &\approx -\cos x.
\end{align*}
\]  

(4.36)
Then the solutions of (32) are

\[
C_1 \approx -\frac{\hbar kR}{\sin kR},
\]
\[
C_2 \approx -\frac{2\mu k}{(2\mu + \lambda) \sin qR} = -2\hbar \frac{c^2}{q^2} \frac{qR}{\sin qR}.
\]  
(4.37)

Using the approaches (34) - (36) and the solutions (28), (29) and (37) we find at \( r = R \) the following expressions:

\[
C_1 \cdot Z^{(i)} = \frac{\hbar}{k} \cot kR \left( \frac{\partial S_3}{\partial \theta} + \phi \frac{1}{\sin \theta} \frac{\partial S_2}{\partial \varphi} \right),
\]
\[
C_2 \cdot Z^{(i)} = -\frac{c^2}{q^2} \frac{\hbar}{q} \cot qR + 2S_2.
\]  
(4.38)

At high frequencies the characteristic attenuation length of elastic waves is small compared to \( R \). Then \( \omega R/(\rho/\mu)^{1/2} \) and \( \omega R/(\rho/(\lambda + 2\mu))^{1/2} \) have a large imaginary part and the factors \( \cot kR \) and \( \cot qR \) become a phase factor, \( \exp \{-i\pi/2\} \), i.e. \(-i\). Finally, the general solution (27) becomes

\[
Z = Z^{(i)}_\varphi \hat{\varphi} + Z^{(i)}_\theta \hat{\theta} + Z^{(i)}_r \hat{r},
\]  
(4.39)

where \( \hat{r}, \hat{\theta}, \) and \( \hat{\varphi} \) are the unit vectors and the correspondent vector components are

\[
Z^{(i)}_\varphi = i \frac{c^2}{\omega} \sin \theta \cos \theta (h_{xx} \cos 2\varphi + h_{yy} \sin 2\varphi),
\]
\[
Z^{(i)}_\theta = -i \frac{c^2}{\omega} \sin \theta (h_{xx} \sin 2\varphi - h_{yy} \cos 2\varphi),
\]
\[
Z^{(i)}_r = i \frac{c^2}{\omega} \frac{1}{\sin^2 \theta} (h_{xx} \cos 2\varphi + h_{yy} \sin 2\varphi).  
\]  
(4.40)

Thus, it was found that the elastic response of the Earth to an arbitrarily polarized plane gravitational wave is a superposition of two plane waves, one longitudinal with the component \( Z^{(i)}_\varphi \) and one transverse with components \( Z^{(i)}_\theta \) and \( Z^{(i)}_r \) which propagate perpendicularly to the sphere surface to a very good approximation. It is interesting to compare the results of the present calculations with those Dyson (1969) for an infinite half-space. Let us assume that a point \( P \) at the sphere surface with coordinates \( (R, \theta, \varphi = 0) \) belongs to the surface of this half-space too. This point is a single point of contact of those surfaces and all others points of sphere are inner points of the infinite half-space. Let us denote a locale Cartesian coordinate system as \( X'Y'Z' \).
and connect its origin of coordinates with point $P$ so that the axis $Z'$ must be directed along the inward normal $N^k$ and inversely to the radius-vector $r$. So it was supposed above a gravitational wave propagates along the $Z$-direction in geocentric coordinate system $XYZ$. Then in coordinate system $X'Y'Z'$ we have the gravitational wave with frequency $\omega$ and wave-vector

$$h' = (\omega/c)[\sin \theta', 0, \cos \theta'],$$

(4.41)

incident at an angle $\theta'$ to the inward normal.

Let the wave be pure $h_+$-polarized (i.e. $h_\times = -h_\varphi$, $a = 1$, $b = 0$). Then Dyson's vector expression (2) that gives complete information concerning the phase, amplitude and direction of the seismic displacement induced at the Earth's surface by a gravitational wave becomes

$$Z^i = i h_+ \frac{c_t}{\omega} \sin \theta' [\cos \theta', 0, -\frac{c_i}{c_t} \sin \theta'].$$

(4.42)

Thus all the instruments in a seismic array should respond coherently to a gravitational wave, provided that the extent of array is small compared with the free-space wavelength $(2\pi c/\omega)$.

On the basis of $\varphi = 0$, $h_x = h_\varphi = h_\times = 0$, and $h_+ = h_\times = -h_\varphi$ the expression (40) becomes

$$Z^{(1)}_\theta = i h_+ \frac{c_t}{\omega} \sin \theta \cos \theta,$$

$$Z^{(1)}_\varphi = 0,$$

$$Z^{(1)}_t = i h_+ \frac{c^2_t}{c_t} \frac{1}{\omega} \sin^2 \theta.$$  

(4.43)

So that $\theta = \theta'$ the formulae (43) are analogous those obtained by Dyson (1969) to the accuracy of last term sign (see formula (42)), because the axis $Z'$ and the radius-vector $r$ are inversely directed. According to (42) at $\theta' = \pi/2$ (i.e. in the case when the GW-source is rising or setting) the amplitude of longitudinal component $Z^\theta = Z^{(1)}_\theta$ is maximal when the transverse component vanishes. The same result can be obtained from (43) at the polar angle $\theta = \pi/2$, i.e. when the seismograph is placed at geographic equator. But there is an disagreement between Dyson's result and our one obtained above. Though, our result compares favourably with that previously calculated by Dyson (1969) in preserving of dependence of longitudinal and transverse wave magnitudes upon coordinate angles $\theta$ and $\varphi$. Let us denote so called coefficients of "coordinate attenuation" as $C_{h+}, C_{h\times}, C_{h\varphi}$ and $C_{\theta+}, C_{\theta\times}, C_{R\times}$ which correspond to gravitational wave amplitudes of linear polarization of $h_+$ and $h_\times$. The coefficients show an amplitude discrepancy between the different responses induced by a gravitational wave in a seismic array.
Then according to (40) we have

\[ C_{\phi+} = \sin \theta \cos \theta \cos 2\varphi, \]

\[ C_{\varphi+} = \sin \theta \sin 2\varphi, \]

\[ C_{R+} = \sin^2 \theta \cos 2\varphi; \]

\[ C_{\delta X} = \sin \theta \cos \theta \sin 2\varphi, \]

\[ C_{\varphi X} = \sin \theta \cos 2\varphi, \]

\[ C_{R X} = \sin^2 \theta \sin 2\varphi. \quad (4.44) \]

The values of those coefficients calculated in virtue of the table 3 about the local seismic array are tabulated. It is suggested that the angle of incidence of gravitational wave is optimal.

According to the table 5 those coefficients of "coordinate attenuation" differ by less than 16%.

The discrepancy is not essentially for the present investigation.
Table 5. The values of coefficients of "coordinate attenuation" which correspond to gravitational wave amplitudes of linear polarization of $h_+$ and $h_\times$ calculated in virtue of the table 4 about the local seismic array. It is suggested that the angle of incidence of gravitational wave is optimal.

<table>
<thead>
<tr>
<th>Code</th>
<th>$C_{e+}$</th>
<th>$C_{e\times}$</th>
<th>$C_{R+}$</th>
<th>$C_{R\times}$</th>
<th>$C_{o+}$</th>
<th>$C_{o\times}$</th>
<th>$C_{R\times}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GSC</td>
<td>1.000</td>
<td>0.925</td>
<td>0.999</td>
<td>0.945</td>
<td>1.000</td>
<td>0.915</td>
<td></td>
</tr>
<tr>
<td>ISA</td>
<td>0.923</td>
<td>0.959</td>
<td>0.911</td>
<td>0.968</td>
<td>0.915</td>
<td>0.945</td>
<td></td>
</tr>
<tr>
<td>PAS</td>
<td>0.920</td>
<td>0.970</td>
<td>0.960</td>
<td>0.962</td>
<td>0.947</td>
<td>0.974</td>
<td></td>
</tr>
<tr>
<td>RPV</td>
<td>0.903</td>
<td>0.980</td>
<td>0.958</td>
<td>0.962</td>
<td>0.940</td>
<td>0.988</td>
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</tr>
<tr>
<td>SBC</td>
<td>0.848</td>
<td>0.999</td>
<td>0.875</td>
<td>0.999</td>
<td>0.867</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>USC</td>
<td>0.913</td>
<td>0.974</td>
<td>0.957</td>
<td>0.964</td>
<td>0.943</td>
<td>0.979</td>
<td></td>
</tr>
</tbody>
</table>

5. An algorithm of optimal seismic data processing

Let us assume that for registration of seismic vibrations $U(x',y',z',t)$ it is utilized the seismic array composed from $M$ elements setting on free surface $Z' = 0$ in coordinate system of $X'Y'Z'$. Then the output of this array after inverse filtering is

$$V(t) = \{V_k(t) = U(x'_k, y'_k, z'_k)\}, \quad k = 1, 2, 3, \ldots M.$$  \hspace{1cm} (5.45)

It is useful to represent responses of each seismic array channel as a combination of "signal + noise" at the input of each seismic meters

$$U(x',y',z',t) = S(x',y',z',t) + N(x',y',z',t),$$ \hspace{1cm} (5.46)

where $S(x',y',z',t)$ is a signal wave induced by gravitational waves, and $N(x',y',z',t)$ is a seismic noise of different nature.

Then the substitution of expression (46) to (45) gives next result:

$$V(t) = S(t) + N(t),$$ \hspace{1cm} (5.47)
where $S(t) = \{S_k(t) = S(x_k', y_k', z_k')\}$, $N(t) = \{N_k(t) = N(x_k', y_k', z_k')\}$.

Let us take for definiteness the following form of gravitational wave signal:

$$h(t) = h_0 \cos \omega_p t, \quad |t| \leq \hat{t};$$
$$0, \quad |t| > \hat{t}. \quad (5.48)$$

where $\omega_p$ and $h_0$ are the angular frequency and dimensionless amplitude of GW-burst whose duration obeys an equation $\omega_p \hat{t} \approx 2\pi$.

Suppose that the wave vector of incident gravitational wave is in $Z'X'$ plane with angle $\theta'$ to $Z'$ axis. Let the wave be pure right-handed circularly polarized. Then according to Dyson's expression (2) we have in the long-wave approximation that the signal vector $S^j$, $j = 1, 2, 3$ is

$$S^j(t) = \{S^j_w(t), S^j_v(t), S^j_r(t)\} \quad (5.49)$$

where

$$S^j_w(t) = \frac{1}{2} h_0 \frac{s}{\omega_p} \sin \theta' \cos \theta' \cos (\omega_p t - \pi/2),$$
$$S^j_v(t) = \frac{1}{2} h_0 \frac{s}{\omega_p} \sin \theta' \cos \omega_p t,$$
$$S^j_r(t) = \frac{1}{2} h_0 \frac{s^2}{\omega_p v} \sin^2 \theta' \cos (\omega_p t + \pi/2), \quad (5.50)$$

whose $(X', Y')$-components are transverse $S^T_j$ at $j = 1, 2$ and $Z'$-component is longitudinal $S^L_j$ at $j = 3$. The mass-elements on the Earth’s surface according to formulae (49), (50) describe elliptical orbits with the major axes in the $Y'$-direction. If the incident wave is left-circularly polarized, only the sing of $Y'$-component in formulae (49), (50) has to be reversed. The response for a linearly polarized wave could be obtained by taking a linear superposition of the two circularly polarized responses.

Therefore, at $\theta' = \pi/2$ when the gravitational wave is running along free surface $Z' = 0$ the longitudinal component of the Earth’s response induced by the linearly $h_+ \cdot$-polarized gravitational wave (i.e. $h(t) = h_+ (t)$) is maximum and the expression (50) becomes

$$S^j_w(t) = 0; \quad S^j_v(t) = 0;$$
$$S^j_r(t) = h_0 \frac{s^2}{\omega_p v} \cos (\omega_p t + \pi/2). \quad (5.51)$$

We will assume farther that the velocities of longitudinal ($v = c_0$) and transverse ($s = c_1$) seismic waves are equal to 4.5 km/s.
The optimal data processing of vector signal (47) that ensures maximum of signal-noise ratio at infinite observing interval \(-\infty < t < \infty\) is the linear functional (Gusev et al., 1990; Gusev and Kravchuk, 1994; Kravchuk et al., 1995)

\[
F(t) = K_0 \sum_{k=1}^{M} \int_{-\infty}^{+\infty} V_k(t) R_k(t + \tau) \, d\tau = \frac{K_0}{2\pi} \sum_{k=1}^{M} \int_{-\infty}^{+\infty} \hat{V}_k(j\omega) \hat{R}_k(j\omega) \exp(-j\omega t) \, d\omega, \tag{5.52}
\]

where \(K_0\) is the arbitrary scale factor; \(\hat{V}_k(j\omega)\) and \(\hat{R}_k(j\omega)\) are the spectral representation of \(V_k(t)\) and \(R_k(t)\).

At stationary and stationary connected noises \(\{N_k(t)\}\) the vector

\[
R = \{\hat{R}_1, \hat{R}_2, \ldots, \hat{R}_M\}
\]

is calculated from next simultaneous of equations

\[
\sum_{\mu=1}^{M} G_{k\mu}(j\omega) \hat{R}_\mu(j\omega) = \hat{S}(j\omega), \quad k = 1, \ldots, M, \tag{5.53}
\]

where \(G_{k\mu}(j\omega)\) at \(k \neq \mu\) are the cross-spectra between seismic noises in \(k\)-th and \(\mu\)-th channels of seismic array, and \(G_{kk}(j\omega)\) at \(k = \mu\) is the power spectrum of seismic noise in \(k\)-th channel of seismic array; \(\hat{S}(j\omega)\) is the spectral representation of GW-response \(S(t)\).

The signal-noise ratio provided of the optimal proceeding within the frequency range of \(\sim 0.01 - 0.1\) Hz is (Gusev and Kravchuk, 1994)

\[
\left\{ \frac{S}{N} \right\}_M \approx \frac{1}{2\pi} \sum_{k=1}^{M} \int_{-\infty}^{+\infty} \hat{S}(j\omega) \hat{R}_k(j\omega) \, d\omega \approx \frac{1}{\pi} \sum_{k=1}^{M} \int_{0.01\pi}^{0.03\pi} \hat{S}(j\omega) \hat{R}_k(j\omega) \, d\omega. \tag{5.54}
\]

Let us write the expressions for the signal-noise ratio \((S/N)\) at \(M = 2\) and \(M = 3\).

**The case 1:** the two-channel seismic array, (i.e. at \(M = 2\)).

In this case the expression (53) is the system of two leaner algebraic equations relatively unknown quantities \(\hat{R}_1(j\omega)\) and \(\hat{R}_2(j\omega)\) and signal-noise ratio becomes:

\[
\left\{ \frac{S}{N} \right\}_2 \approx \frac{1}{\pi} \int_{0.01\pi}^{0.03\pi} \frac{\hat{S}(j\omega) |^2 (G_{11}(\omega) + G_{22}(\omega) - 2\text{Re}[G_{12}(j\omega)])}{G_{11}(\omega) G_{22}(\omega) (1 - |\gamma_2(j\omega)|^2)} \, d\omega, \tag{5.55}
\]
where $\gamma_{12}(j\omega)$ is the complex coherence function between seismic noises in first and second channels of array, whose square modulus magnitude is defined as

$$|\gamma_{12}(j\omega)|^2 = \left| \frac{G_{12}(j\omega)}{G_{11}(\omega) G_{22}(\omega)} \right|^2,$$  \hspace{1cm} (5.56)

where the notation $\text{Re} \{ \ldots \}$ is the real part of complex number.

If the seismic noises are uncorrelated then $|\gamma_{12}(j\omega)|^2 = 0$, $\text{Re} \{G_{12}(j\omega)\}$ goes to zero and the signal-noise ratio $(S/N)_2$ is equal to sum of two signal-noise ratios in outputs of two arbitrary seismographs provided an one-channel seismic data processing. For the strong cross-correlation between seismic noises the magnitude $|\gamma_{12}(j\omega)|^2$ goes to unity and the signal-noise ratio $(S/N)_2$ could be strongly increased, on principle.

The case 2: the three-channel seismic array, (i.e at $M = 3$).

In this case the expression for signal-noise ratio is more complicated

$$\left\{ \frac{S}{N} \right\}_3 \approx \frac{1}{\pi} \int_{0}^{2\pi} \frac{|\hat{S}(j\omega)|^2 \sum_{k=1}^{3} D_k(j\omega)}{\text{Det}(j\omega)} \text{d}\omega,$$  \hspace{1cm} (5.57)

where:

$$D_1(j\omega) = G_{11}(\omega) G_{22}(\omega) \left( 1 - |\gamma_{12}(j\omega)|^2 \right),$$

$$D_2(j\omega) = G_{11}(\omega) G_{33}(\omega) \left( 1 - |\gamma_{13}(j\omega)|^2 \right),$$

$$D_3(j\omega) = G_{22}(\omega) G_{33}(\omega) \left( 1 - |\gamma_{23}(j\omega)|^2 \right),$$

$$D_4(j\omega) = -2G_{11}(\omega) \text{Re} \{G_{23}(j\omega)\} + G_{22}(\omega) \text{Re} \{G_{13}(j\omega)\} + G_{33}(\omega) \text{Re} \{G_{11}(j\omega)\},$$

$$D_5(j\omega) = 2(\text{Re} \{G_{12}(j\omega) G_{23}(j\omega)\} + \text{Re} \{G_{13}(j\omega) G_{23}(j\omega)\} + \text{Re} \{G_{13}(j\omega) G_{13}(j\omega)\}),$$

$$\text{Det}(j\omega) = G_{11}(\omega) G_{22}(\omega) G_{33}(\omega) \times$$

$$\times \left( 1 - |\gamma_{12}(j\omega)|^2 - |\gamma_{13}(j\omega)|^2 - |\gamma_{23}(j\omega)|^2 \right) +$$

$$+ 2\text{Re} \{G_{12}(j\omega) G_{13}(j\omega) G_{23}(j\omega)\},$$

and the notation * denotes a complex conjugation.

For the case of four-channel (i.e at $M = 4$) seismic array the system of equations (53) is lightly solved by numerical methods on the selected frequency set.

The unknown values of cross-spectra $G_{km}(j\omega)$ between seismic noises of $k$-th and $m$-th channels of seismic array and seismic noise power spectrum $G_{kk}(j\omega)$ in $k$-th channel have been estimated by the weighted overlapped segment averaging (Carter et al., 1973) on the basis of FFT-technique to the digital row seismic data. The spectral frequency resolution $\Delta f_{\text{res}}$ was about of $1.2 \times 10^{-4}$ Hz.

According to the theory of optimal $M$-channel filtration (Kravchuk et al., 1995) the minimal detectable dimensionless magnitude of GW burst $h_{\text{min}}$ can be straightforwardly calculated from the condition $(S/N)_{M} \approx 1$ for any seismic array composed from $M$ elements.
6. Limits on an intensity of GW-bursts obtained on the basis of seismic array data processing

The upper limits on an intensity of short gravitational wave bursts is obtained from next two seismic array data records covered half-year time interval.

In first case it was utilized the seismic array composed from three seismometers placed at next seismic station: Pasadena, Los Angeles and Rancho Palos Verge. This seismic array have got almost a linear configuration.

In second case it was utilized the seismic array composed from four seismometers placed at next seismic station: Pasadena, Isabella, Santa Barbara and Goldston. This seismic array have got approximately a crosswise configuration. Parameters of those seismic stations is presented above in the table 4.

After the calculation of spectral estimates of cross-spectra $\hat{\chi}_{hh}(j\omega)$, power spectra $\hat{\chi}_{bb}(j\omega)$ and magnitude-squared coherence (MSC) function $|\hat{\chi}_{bb}(j\omega)|^2$ the numerical values of signal-noise ratios $(S/N) |_k$ was obtained at $k = 3, 4$, and then the upper limits on an intensity of short gravitational wave bursts have been straightforwardly defined.

Table 6. Main parameters of used seismic array and upper limit estimates on an intensity of short GW-bursts established from seismic array data processing and integrated over low-frequency range of $\sim 0.01 - 0.1$ Hz.

<table>
<thead>
<tr>
<th>Parameters of seismic arrays</th>
<th>3-th channels seismic array</th>
<th>4-th channels seismic array</th>
</tr>
</thead>
<tbody>
<tr>
<td>Configuration</td>
<td>PAS-USC-RPV</td>
<td>PAS-ISA-SBC-GSC</td>
</tr>
<tr>
<td>Type of channels</td>
<td>LHE, horizontal</td>
<td>LHZ, vertical</td>
</tr>
<tr>
<td>Form of array</td>
<td>linear</td>
<td>crosswise</td>
</tr>
<tr>
<td>Range of MSC</td>
<td>$0.23 \leq</td>
<td>\hat{\chi}</td>
</tr>
<tr>
<td>Polarization of GW Upper limits:</td>
<td>$\hat{\chi}_x$</td>
<td>$\hat{\chi}_x$</td>
</tr>
<tr>
<td>- from array</td>
<td>$(6.6 \pm 2.5) \times 10^{-14}$</td>
<td>$(1.0 \pm 0.1) \times 10^{-14}$</td>
</tr>
<tr>
<td>- from single seismographs</td>
<td>$(1.2 \pm 0.7) \times 10^{-12}$</td>
<td>$(1.4 \pm 0.3) \times 10^{-13}$</td>
</tr>
</tbody>
</table>
We must note that all used digital seismic traces contained seismic events produced by small magnitude earthquakes \((M_s \leq 4)\).

From three elements seismic array data processing the MSC-functions were within the range of \(0.23 \leq \max\{|\tilde{\gamma}_k(j\omega_p)|^2\} \leq 0.94\). From four elements one the MSC-functions lied within the range of \(0.32 \leq \max\{|\tilde{\gamma}_k(j\omega_p)|^2\} \leq 0.85\).

Table 6 summarises the main parameters of used seismic array and upper limit estimates on an intensity of short GW-bursts established from 3-th and 4-th components seismic array data processing and integrated over low-frequency range of \(\sim 0.01 - 0.1\) Hz. For a comparison it was presented the results of upper limit estimations obtained on the basis of single seismograph data processing (i.e at using of one-channel optimal data processing).

7. Results

On the basis of half-year seismic data record within the low-frequency range of \(\sim 0.01 - 0.1\) Hz integrated estimates of upper limit on the gravitational wave burst intensity are found by utilising three and four component adaptive seismic array. The best value of upper limit is \(h_{\text{opt}} \cong (1.0 \pm 0.1) \times 10^{-16}\) in terms of metric perturbations. To obtain this result the vertical measuring channel of four-component seismic array (Pasadena - Isabella - Santa Barbara - Goldston) is utilised. The correspondent value in terms of ground displacements \(\Delta y\) is about of \(1.4 \times 10^{-6}\) cm at the middle frequency \(f_p = 0.05\) Hz. According to Aki, Richards et. al. (1983) the average power spectrum of seismic noise in this frequencies is about of \(10^{-7}\) cm \(\cdot\) Hz\(^{-1/2}\) at seismically quiet location. Thence the standard deviation of seismic noise at the frequency band about of \(\sim 0.1\) Hz must be about of \(\sim 3.0 \times 10^{-6}\) cm. The value resulted from our calculation is less about the factor two. Meanwhile the ground displacement magnitude \(\Delta y_{\text{opt}}\) calculated on the basis of most optimistic astrophysical forecasts of \(h_{\text{opt}} \sim 10^{-16}\) has to be about \(10^{-10}\) cm. Thus the expected response of the Earth's surface to GW-bursts is occurred to be about a factor of \(10^3\) smaller the value which could be detected. According to previous Dyson's estimations (Dyson, 1969) the discrepancy would have been larger about a factor of \(10^3\).

Let us execute an equivalent conversion of \(h_{\text{opt}}\) to the spectral density of gravitational radiation flux \(J_\nu\). For plane \(h_\perp\)-polarised gravitational wave propagating alone the axis \(Z\) at least one component of pseudotensor \(\mathbf{t}^{\mu\nu}\) is not vanished. It is \(\mathbf{t}^{03}\) component that can be expressed by the following formula (Landau and Lifshits, 1971)

\[
\mathbf{t}^{03} = w = \frac{c^3}{16\pi G} \left\{ \frac{1}{4} \left( \dot{\mathbf{h}}^2_{zz} - \dot{\mathbf{h}}^2_{\nu\nu} \right) \right\} = \frac{c^3}{16\pi G} \dot{\mathbf{h}}^2_{zz}
\]  

(7.58)
It presents the power density or density of radiation flux in terms of \( \text{erg}/(\text{cm}^2 \cdot \text{s}) \); in (58) \( G = 6.670 \times 10^{-8} \text{ cm}^3/(\text{g} \cdot \text{s}) \) is the gravitational constant. For short quasi-harmonic pulse obeying by the condition of \( \omega_0 \tau_s = 2\pi \) at \( |t| \leq \tau_s/2 \) formula (58) after time averaging yields

\[
\omega = \frac{c^3}{32\pi G \omega_0^2 h_0^2}.
\]  

(7.59)

The burst energy \( I_b \) transferred through the unit square one can write down as:

\[
I_b = \omega \tau_s = \frac{c^3}{32\pi G \omega_0^2 h_0^2 \tau_s} = \frac{c^3}{16G} \omega_0 h_0^2 = \frac{\pi c^3}{8G} \nu_0 h_0^2.
\]  

(7.60)

At the same time \( I_s \) can be interpreted as the spectral intensity of power density (i.e., power density per unite of frequency band \( \Delta \nu = 1/\tau_s \)): \( I_s = I_s(\nu) = \omega/\Delta \nu = \omega \tau_s \) in terms of \( \text{erg}/(\text{cm}^2 \cdot \text{s} \cdot \text{Hz}) \).

Numerical calculations according to formula (60) at the frequencies of \( \nu_1 = 0.01 \text{ Hz} \) and \( \nu_2 = 0.02 \text{ Hz} \) give the following estimations:

\[
I_s(\nu_1) = 1.6 \times 10^9 \text{ erg}/(\text{s} \cdot \text{cm}^2 \cdot \text{Hz}),
\]

\[
I_s(\nu_2) = 3.2 \times 10^8 \text{ erg}/(\text{s} \cdot \text{cm}^2 \cdot \text{Hz}).
\]  

(7.61)

We took these frequencies for comparison the upper limit estimate of stochastic gravitational wave background given by Boughn and Kuhn (1984) from observations of terrestrial mode oscillations. One can see from the table 3 that our values (61) differ from the ones obtained by Boughn and Kuhn (1984) by a factors of \( \approx 50 \) at \( \nu = \nu_1 = 0.01 \text{ Hz} \) and \( \approx 10^{-2} \) at \( \nu = \nu_2 = 0.02 \text{ Hz} \), respectively.

Thus we may take as a dominant estimate of upper limit on spectral intensity of power density for GW-bursts (in the frequency region of about 0.01 - 0.1 Hz) the value:

\[
I_s < 1.6 \cdot 10^8 \text{ erg}/(\text{s} \cdot \text{cm}^2 \cdot \text{Hz}).
\]  

(7.62)

A useful control mark with which one can check this limit is the matter density for closure Universe, i.e. \( \rho_c \approx 2 \cdot 10^{-30} \text{ g/cm}^3 \) (assuming a Hubble constant of 100 km/(s \cdot Mpc)). Following from that the estimation of gravitational radiation flux \( F_\nu \) can be calculated from the evident relation \( \nu F_\nu = \rho_c c^3 \):

\[
F_\nu = \rho_c c^3 / \nu \approx 2 \cdot 10^{-30} \times 27 \cdot 10^{30} / 10^{-2} \approx 5.4 \cdot 10^4 \text{ erg}/(\text{s} \cdot \text{cm}^2 \cdot \text{Hz}).
\]  

(7.63)

Any gravitational wave flux substantially larger then the value of \( \nu F_\nu \) conflicts with observations which imply the Universe is approximately open. According to various scenarios for the generation of gravitational radiation (Carr, 1980) the value of \( \nu F_\nu \) could be as large as \( 10^{-2} \rho_c c^3 \).
over periods in the range from $10^{-3}$ s to $10^5$ s. It is clear that the limit (62) derived in this paper do not constrain accepted models of the Universe.

It is useful to note that the limit (62) has been established on the local adaptive seismic array having the fixed spatial configuration. Modification of location and configuration could be in general to change the value of upper limit on spectral intensity of power flux density from GW-bursts. Rigorous experimental solution of upper limit problem supposes to utilize a global net of seismic arrays having an optimal configuration with respect to spherical coordinates (Kochik and Rudenko, 1995).

We thank O.E.Starovoit for presenting to us the digital data from "TERRASCOPE" seismic net, seismic net.

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