Dynamics of inhomogeneous models near a singularity in classical and quantum cosmology

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Chapter 1

Introduction: Generalized Kasner Variables in inhomogeneous cosmological models

1. Introduction

The problem of the cosmological singularity remains to be one of the most important problems in modern theoretical physics. In studying the singularity we meet two major difficulties. The first one is that we do not know the laws steering physics in such a region and have to use significant extrapolation. In particular, the number of modern unified theories predict that the dimension of the Universe exceeds the one we normally experience at the macroscopic level [14]. It is assumed that at present additional dimensions are hidden, for they are compactified to Planckian size, and they do not display themselves in macroscopic and even microscopic processes. However, the situation might be changed when we consider the very beginning of the evolution of our Universe where the Universe size could approach the Planckian scale. Therefore, in the early Universe additional dimensions, if they exist, must not be different from ordinary dimensions and should be taken into account. This enables us to consider more general than Einstein's one multidimensional theories of gravity [37] in order to inquire into the nature and properties of singularities.

The second difficulty appears from the fact that the singularity, as is widely accepted, requires quantum gravity to provide its exhaustive description. We do not have any reasonable theory of such a kind yet and, therefore, in the absence of such a theory, of essential interest represents the possibility of constructing and studying the models which are from one side sufficiently complex to describe real properties of gravitational fields near the singularity, and, from the other side, sufficiently simple to admit their quantum consideration. We stress that in order to construct and choose such models of principal significance has their classical investigation being carried out in the general inhomogeneous case.

Studying the behavior of inhomogeneous cosmological models near the singularity from the classical point of view was initiated by V.A. Belinsky et al. more than 20 years ago [5] in the case $D = 4$ and was continued in Ref. [6, 10] for the multidimensional case. Properties of inhomogeneities of the metric based on the general solutions was considered first in Refs. [21, 24] for the case $D = 4$ and the multidimensional generalization was given in Ref. [29]. We note also that less general inhomogeneous models have been considered in Ref. [3]. Quantization of those inhomogeneous cosmological models was considered in Refs. [25, 30, 31].

As follows from these papers, the main features of the dynamics of an inhomogeneous gravitational field near the singularity can be summarized as follows:

(1). Locally the dynamics of metric functions resembles the behavior of the most general non-diagonal homogeneous models. In other words, near the singularity the gravitational field becomes quasi-homogeneous one, i.e., acquires a large-scale character.

(2). In the vicinity of a singularity a scalar field is the only kind of matter effecting the dynamics of the metric.

The best way to understand the first fact is to consider the most simple isotropic model. Let us consider the $D-$ dimensional interval in the form (Throughout these lectures we do not specify the number of dimensions D, unless otherwise is noted.)

$$ds^2 = dt^2 - a^2 d\mathbf{x}^2 \tag{1.1}$$

with the spatial interval being corresponding to the flat space. The gravitational part of the action is taken in the usual form

$$I_g = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}R \tag{1.2}$$

(for the sake of simplicity in what follows we put $k = 1$). Then the Einstein equations can be read off

$$R_k^i - \frac{1}{2}\delta_k^i R = T_k^i\ , \tag{1.3}$$

where for the stress-energy tensor we adopt the expression in the form of a perfect fluid

$$T_{ik} = (p+\varepsilon)\,u_i u_k - p g_{ik}\ . \tag{1.4}$$

For the metric (1.1) the Einstein equations give

$$\begin{aligned} R_0^0 &= -n\frac{\ddot a}{a} = \frac{1}{2}(\varepsilon + np) \\ R_0^0 - \frac{1}{2}R &= \frac{n(n-1)}{2}\left(\frac{\dot a}{a}\right)^2 = \varepsilon\ , \end{aligned} \tag{1.5}$$

where $n = D - 1$ is the number of spatial dimensions and the dots, as usual, stands for derivative with respect to cosmological time t. From these equations one obtains that close to the singularity (as $t \to 0$) the scale factor behaves as $a \sim t^{\alpha}$ with the exponent

$$\alpha = \frac{4\varepsilon}{4\varepsilon + (n-1)(\varepsilon + np)} .$$

Inhomogeneities of the metric can be considered within this model as small perturbations δg_{ik} of the isotropic metric (1.1). As is known, in cosmology the only natural scale is the horizon size l_h (which can be used to measure the distance from the singularity)

$$l_h = a(t) \int_0^t \frac{dt'}{a(t')} . \tag{1.6}$$

Using this scale inhomogeneities of the metric may be divided into large-scale ($l_i \gg l_h$) and small-scale ($l_i \ll l_h$) ones. From (1.6) we find that the horison size varies with time as a linear fnction $l_h \sim t$, whereas scales of inhomogeneities behave as the scale factor does $l_i \sim t^{\alpha}$ (as $t \to 0$) and what is important, $\alpha < 1$. Thus, it is clear that an arbitrary inhomogeneous field becomes large scale sufficiently closely to the singularity. We note that more rigorous consideration (in the nonlinear case) does not change qualitatively this property.

Since inhomogeneities are large scale, there are no effects connected with the propagating of gravitational waves, etc., and this would mean that near the singularity inhomogeneities become passive. Therefore, the dynamics of the field may be approximately described by the most general homogeneous model depending parametrically upon the spatial coordinates. Note, however, that the homogeneous model would appear to be in a general non-diagonal form.

To understand the second fact we have to consider more complicated model, for the isotropic Universe cannot exist without matter. Let us consider the simplest homogeneous anisotropic model (the so-called Kasner model) with the metric being given by the interval

$$ds^2 = dt^2 - \sum a_A^2 \left(dx^A\right)^2 . \tag{1.7}$$

Now let us assume that matter is negligible as compared with pure gravitational degrees of freedom. Then the Einstein equations (1.3) are [34]

$$\begin{aligned} R_0^0 &= \Sigma \frac{\ddot{a}_i}{a} = 0 , \\ R_i^i &= -\frac{1}{V}\left(V\frac{\dot{a}_i}{a_i}\right)^{\cdot} = 0 , \end{aligned} \tag{1.8}$$

where $V = \sqrt{-g} = \prod a_i$ is the volume element. From these equations we obtain the well known Kasner solution $a_i = a_i^0 t^{s_i}$ with exponents satisfying the identities $\sum s_i = \sum s_i^2 = 1$. Hence, we find the estimate for behavior of curvature terms $R_k^i \sim t^{-2}$.

The simplest way to obtain similar estimates for the stress-energy terms is to consider the perfect fluid having pure potential motions. In this case the velocity u_i can be presented in the form $u_i = \partial_i\phi / \left(\phi_{;k}\phi^{;k}\right)^{1/2}$ and the stress-energy tensor (1.4) can be obtained from the variation principle with the Lagrangian density being

$$L = \frac{1}{2}\sqrt{-g}\left(\phi_{;k}\phi^{;k}\right)^{\mu} \tag{1.9}$$

and with the exponent μ depending upon the equation of state as follows $\mu = \frac{p+\varepsilon}{2p}$. The energy density ε and the pressure p are expressed via the field ϕ by the expressions $\varepsilon = (\mu - \frac{1}{2})\left(\phi_{;k}\phi^{;k}\right)^{\mu}$ and $p = \frac{1}{2}\left(\phi_{;k}\phi^{;k}\right)^{\mu}$ respectively. The equation of motion for the field function ϕ as it follows from (1.9) is

$$\frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ik}\left(\phi_{;\sigma}\phi^{;\sigma}\right)^{\mu-1}\partial_k\phi\right) = 0$$

which in the case of the Kasner model takes the form

$$\left(V\left(\dot{\phi}\right)^{2\mu-1}\right)^{\cdot} = 0$$

and has the solution $T^i_k \sim \varepsilon \sim t^{-2k}$ where $k = \frac{\varepsilon+p}{2\varepsilon}$. Thus, one can see that for the equation of state satisfying the inequality $p < \epsilon$ we have $k < 1$ and only for the limiting case $p = \epsilon$ do both terms in (1.3) turn out to be of the same order and the expression (1.9) coincides with the Lagrangian of the ordinary scalar field.

In the general inhomogeneous case the metric functions can be divided into two groups of variables. Near the singularity the first group has behavior like a set of coupled scalar fields while residual variables behave as a set of vector fields and can be neglected in a leading order (in the same manner as it happens for the matter having an equation of state $\epsilon > p$) [34, 29]. In order to make such division in an explicit form it turns out to be convenient to use the so-called Kasner-like parametrization of the dynamical functions [24, 29].

2. Generalized Kasner model, Generalized Kasner Variables

The most simple way to introduce the generalised Kasner variables is to consider the generalized Kasner solution [34]. Let us consider the canonical formulation of gravity. The $D-$dimensional interval can be represented in the form

$$ds^2 = N^2dt^2 - g_{\alpha\beta}\left(dx^\alpha + N^\alpha dt\right)\left(dx^\beta + N^\beta dt\right),$$

where N and N^α are the lapse function and the shift vector respectively which play the role of Lagrangian multipliers. The basic variables are the Riemann metric components $g_{\alpha\beta}$ and a scalar field ϕ specified on the n-dimensional spatial manifold S, and their conjugate momenta $\Pi^{\alpha\beta} = \sqrt{g}(K^{\alpha\beta} - g^{\alpha\beta}K)$ and Π_ϕ, where $\alpha = 1, ..., n$ and $K^{\alpha\beta}$ is the extrinsic curvature of S.

For the sake of simplicity we shall consider S to be compact, i.e., $\partial S = 0$ (one may consider S to be an n-dimensional sphere though this has no significance for our investigation, for our analysis will be mostly of local character). The action has the following form in the Planck units (see, for example, [38]):

$$I = \int_S \left(\Pi^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha \right) d^n x dt, \tag{2.10}$$

with the Hamiltonian and momentum constraints being given by the expressions

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \Pi^\alpha_\beta \Pi^\beta_\alpha - \frac{1}{n-1} (\Pi^\alpha_\alpha)^2 + \frac{1}{2} \Pi^2_\phi + g(W(\phi) - R) \right\}, \tag{2.11}$$

$$H^\alpha = -2\Pi^{\alpha\beta}_{|\beta} + g^{\alpha\beta} \partial_\beta \phi \Pi_\phi, \tag{2.12}$$

where the scalar field potential is

$$W(\phi) = \frac{1}{2} \left\{ g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right\}. \tag{2.13}$$

2.1. Generalized Kasner model

A generalized Kasner solution is realised under the assumption that we can neglect in (2.10) the potential terms in comparison with the kinetic terms

$$\frac{1}{\sqrt{g}} T \sim \frac{1}{g} (\Pi^\alpha_\beta, \Pi_\phi) \gg (W, R), \tag{2.14}$$

where $\sqrt{g} T$ denotes the first three terms in (2.11). Then, using the gage $N = 1$, $N^\alpha = 0$ we , from (2.10), obtain the Einstein equations in the form

$$\begin{gathered}
\tfrac{\partial}{\partial t} \Pi^\alpha_\beta = 0 \ , \\
\tfrac{\partial}{\partial t} g_{\alpha\beta} = \tfrac{2}{\sqrt{g}} \left(\Pi_{\alpha\beta} - \tfrac{1}{n-1} g_{\alpha\beta} \Pi^\sigma_\sigma \right) \ , \\
\tfrac{\partial}{\partial t} \Pi_\phi = 0 \ , \\
\tfrac{\partial}{\partial t} \phi = \tfrac{1}{\sqrt{g}} \Pi_\phi \ ,
\end{gathered} \tag{2.15}$$

which should be completed by the constraint equations

$$H = 0 \ , \ H^\alpha = 0 \ . \tag{2.16}$$

This system of equations gives the so-called generalised Kasner solution constructed first by Lifshitz and Khalatnikov in 1963 [34]:

$$
\begin{aligned}
g_{\alpha\beta} &= \sum_i t^{2s_i} \ell^i_\alpha(x)\,\ell^i_\beta(x)\,, \\
\Pi^\alpha_\beta &= \sum_i \pi^i(x)\, L^\alpha_i(x)\,\ell^i_\beta(x)\,, \\
\phi &= \phi_0(x) + q(x)\ln t\,, \\
\Pi_\phi(x) &= \frac{\sqrt{2}}{n-1} q(x) \sum_i \pi^i(x)\,,
\end{aligned}
\tag{2.17}
$$

where $L^\alpha_i(x)$ are vectors dual to $\ell^i_\alpha(x)$, ($L^\alpha_i \ell^i_\beta = \delta^\alpha_\beta$) and the Kasner exponents $s_i(x)$ are expressed via eigenvalues π^i of the momentum matrix as follows

$$
s_i(x) = 1 - (n-1)\frac{\pi^i(x)}{\sum \pi} \tag{2.18}
$$

and satisfy the identity $\sum s_a = 1$. The momentum constraint (2.16) reduces the number of arbitrary functions contained in the Kasner vectors $\ell^i_\alpha(x)$ down to $n^2 - n$, whereas the Hamiltonian constraint imply the additional restriction on the Kasner exponents

$$
\sum s_a^2 + q^2 = 1\,. \tag{2.19}
$$

Thus, the Kasner exponents run over the range $-\frac{n-2}{n} \le s_a \le 1$.

Now we are ready to understand the meaning of the conditions of applicability of the generalised Kasner model which are given by the inequalities (2.14). As it follows from (2.17) $g = \left(\sum \pi^i\right)^2 t^2$ and hence, for the scalar field ϕ one finds that the left hand side of the inequalities (2.14) has the order $\frac{1}{g}\Pi^2_\phi \sim t^{-2} \sim l_h^{-2}\phi^2$ while the right hand side has the order $W \sim l_i^{-2}\phi^2$ (we note that as $t \to 0$ the potential of the scalar field $V(\phi)$ is negligible in comparison with the terms containing spatial derivatives). Thus, the inequality (2.14) can be re-written as $l_i \gg l_h$. The same meaning has the gravitational part of inequalities (2.14).

In this manner the conditions of the applicability of the generalised Kasner model turns out to be coincided with the conditions that the fields are to be large-scale ones.

2.2. Generalised Kasner variables

¿From (2.17) one can see that in the generalised Kasner model the only evolving functions are the scales of the Kasner vectors. Since, as was shown in Refs. [5, 6], the generalised Kasner solution takes a substantial portion of the evolution of the metric, it is convenient to introduce a Kasner-like parametrization of the dynamical variables [24, 29]. The main idea here is to introduce a new canonical variables which from the very beginning distinguish explicitly the scales of the Kasner vectors. To this end we consider the following representation for the metric components and their conjugate momenta:

$$g_{\alpha\beta} = \sum_a \exp\{q^a\} \ell^a_\alpha \ell^a_\beta\,, \tag{2.20}$$

$$\Pi^\alpha_\beta = \sum_a p_a L^\alpha_a \ell^a_\beta\,, \tag{2.21}$$

where $L^\alpha_a \ell^b_\alpha = \delta^b_a$ $(a, b = 0, ..., (n-1))$, and now the vectors ℓ^a_α contain only $n(n-1)$ arbitrary functions of the spatial coordinates. A further parametrisation may be taken in the form

$$\ell^a_\alpha = U^a_b S^b_\alpha, \qquad U^a_b \in SO(n), \qquad S^a_\alpha = \delta^a_\alpha + R^a_\alpha\ , \tag{2.22}$$

where R^a_α denotes a triangle matrix ($R^a_\alpha = 0$ as $a < \alpha$). Substituting (2.20) - (2.22) into (2.10) we obtain the following expression for the action functional:

$$I = \int_S \left(p_a \frac{\partial q^a}{\partial t} + T^\alpha_a \frac{\partial R^a_\alpha}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha \right) d^n x dt, \tag{2.23}$$

where $T^\alpha_a = 2\sum_b p_b L^\alpha_b U^b_a$ and the Hamiltonian constraint takes the form

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \sum p_a^2 - \frac{1}{n-1} (\sum p_a)^2 + \frac{1}{2}\Pi^2_\phi + V \right\}\ , \tag{2.24}$$

and $V = g(W(\phi) - R)$.

In the case of $n = 3$ the functions R^a_α are connected with just transformations of the coordinate system and may be removed by solving the momentum constraints $H^\alpha = 0$. Indeed, in this case it is convenient to parametrise this matrix by three functions as follows

$$S^a_\alpha = \partial_\alpha y^a(x)\ .$$

Then their momenta will be determined as

$$\mathcal{P}_c = -2\partial_\alpha \left(\sum_{a,b} p_a U_{ab} U_{ac} \partial x^\alpha / \partial y^b \right)$$

and, hence, the momentum constraint takes the most simple form

$$H_\beta = \sum_{a=0}^{2} (p_a \partial_\beta q^a + \mathcal{P}_a \partial_\beta y^a) + \Pi_\phi \partial_\beta \phi$$

which can be easily resolved with respect to $\mathcal{P}_a$ if one considers y^a to be a new coordinates on S.

In the multidimensional case, however, the functions R^a_α contain $\frac{n(n-3)}{2}$ dynamical functions as well and, therefore, we cannot distinguish coordinate functions in the explicit form as in the case $n = 3$.

Now, it is easy to see that the choice of the Kasner-like parametrization simplifies the procedure of constructing the generalized Kasner solution. Indeed, if we now neglect the potential term in (2.23) and put $N^\alpha = 0$, we find that the Hamiltonian does not depend on the scale functions and other dynamical variables contained in Kasner vectors introduced by expressions (2.20) and (2.21).

3. The asymptotic model in the vicinity of a cosmological singularity (the BLK model)

As is well known [5, 6], the Kasner regime (2.17) turns out to be unstable in the general case. This happens due to violation of the condition (2.14) since the potential V contains increasing terms which lead to the replacement of Kasner regimes. Indeed, as was pointed out above in the generalised Kasner model the only evolving variables are the scale functions of the Kasner vectors $a_i = \exp\left(\frac{1}{2}q^i\right) \sim a_i^0 t^{s_i}$ and part of the Kasner exponents can have negative sign. This means that in the directions corresponding to the negative exponents scale functions increase when approaching the singularity.

To distinguish the dependence of the potential upon scale functions let us consider the structure of the potential terms. Let $e^i = e^i_\alpha(x)\, dx^\alpha$ be a basis of one forms on S in which the spatial interval takes the form

$$dl^2 = \delta_{ij} e^i e^j.$$

This basis defines the structure functions $C_{i,jk}$ as

$$de^i = \frac{1}{2} C^i_{jk} e^j \wedge e^k.$$

In terms of these functions the scalar curvature has the form

$$R = \nabla_i C_i - C_i C_i - \frac{1}{2} C_{i,jk} \left(C_{j,ki} + C_{k,ij} - C_{i,ik}\right) \ ,$$

where $\nabla_i = E^\alpha_i \partial_\alpha$ and $C_i = C_{k,ki}$. Thus, choosing this basis in the form $e^i = \exp\left(\frac{1}{2}q^i\right) \ell^i_\alpha(x)\, dx^\alpha$ we find that the structure functions depend upon scale functions q^a in the following manner

$$C_{i,jk} = \lambda_{i,jk} \exp\left(\frac{1}{2}\left(q^i - q^j - q^k\right)\right) \ ,$$

where we distinguished in the explicit form the leading exponential multiplier. In what follows it is convenient to introduce the anisotropy parameters as follows

$$Q_a = \frac{q^a}{\sum q} \, .$$

In terms of these parameters the scale functions take the form $a_i^2 = g^{Q_i}$. Hence, we find that the potential terms can be presented in the form

$$V = \sum_{A=1}^{k} \lambda_A g^{\sigma_A}, \tag{3.25}$$

where λ_A is a set of functions of all dynamical variables and their derivatives and σ_A are linear functions of the anisotropy parameters

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \ , \ \ b \neq c \ . \tag{3.26}$$

Now one can easily see that increasing terms are those ones for which a combination of Kasner exponents of the type (3.26) satisfies the inequality

$$1 + s_a - s_b - s_c < 0.$$

In the case of a general position of the set of the exponents s_a there is only one such term which corresponds to the power (3.26) $\sigma_{0,n-2,n-1}$ with Kasner exponents being given in the increasing order $s_0 \leq ... \leq s_{n-2} \leq s_{n-1}$.

Exercise [5, 6].

Retaining just the increasing term in the potential V *(*$V = \lambda_{0,n-2,n-1} g^{\sigma_{0,n-2,n-1}}$*) to find out the law of the replacement of the Kasner exponents.*

The answer is given by the formulas

$$\begin{gathered} s_0' = \frac{-s_0 - s}{1+2s_0+s} \ , \ s_i' = \frac{s_i}{1+2s_0+s} \ , \\ s_{n-2}' = \frac{2s_1+s+s_{n-2}}{1+2s_0+s} \ , \ s_{n-1}' = \frac{2s_1+s+s_{n-1}}{1+2s_0+s} \ , \\ q' = \frac{q}{1+2s_0+s} \ , \ s = 1 + s_0 - s_{n-2} - s_{n-1} \ < 0, \end{gathered} \tag{3.27}$$

Here we shall use a more simple way. As was shown (e.g., see, Ref. [4]) in the limit $t \to 0$ the maximal value of $\sigma_{o,n-2,n-1}$ tends to zero. This means that in leading order one can use the approximation of "deep oscillations" [4] by means of setting $\sigma_A^{max} = 0$ and considering the replacement of Kasner regimes to occur instantaneous. The model obtained thus we shall call the inhomogeneous BLK (Belinsky - Lifshits - Khalatnikov) model.

The validity of such an approximation can be seen straightforwardly from (2.23) (see Ref. [38] and also Refs. [24, 26]). Indeed, assuming the finiteness of the functions λ_A and considering the limit $g \to 0$ we find that the potential V may be modeled by the potential walls

$$g^{\sigma_A} \to \theta_\infty[\sigma_A(Q)] = \begin{cases} +\infty \ , & \sigma_A < 0, \\ 0 \ , & \sigma_A > 0 \ . \end{cases} \tag{3.28}$$

Thus, in this limit we obtain that the potential V_∞ depends only on scale functions. Then putting $N^\alpha = 0$ we can remove the passive dynamical functions T^α_a and R^α_a from the action (2.23) and get the reduced dynamical system in the form

$$I = \int_S \left(p_a \frac{\partial q^a}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - \lambda \left[\sum p^2 - \frac{1}{n-1} (\sum p)^2 + \frac{1}{2} \Pi_\phi^2 + V_\infty(Q) \right] \right) d^n x dt, \tag{3.29}$$

where λ is expressed via the lapse function as $\lambda = \frac{N}{\sqrt{g}}$.

To conclude this section I would like to point out to the fact that the momentum constraint has disappeared from (3.29). Nevertheless, the momentum constraint $H_\alpha = 0$ still relates all dynamical functions and, thereby, reduces the number of arbitrary functions contained in T^α_a . Therefore the action contains also implicit information about rotation of Kasner vectors. This follows simply from that Kasner vectors , in virtue of (2.22), are functions of all dynamical variables and explicitly contain scale functions and their momenta.

To conclude this section we point out other important inhomogeneous models which can be described within the kasner-like parametrization. Those are the inflationary model which was constructed first by Starobinsky in Ref. [43] and the quasi-isotropic inhomogeneous model which was constructed by Lifshitz and Khalatnikov in Ref. [34]. The first model can be obtained from the action (??) if we impose on initial conditions restrictions in the form

$$R \ll W(\phi) \simeq const.$$

The model obtained describes the inflationary expansion ($g \to \infty$) of an inhomogeneous Universe (see also Ref. [22]). The second model one obtains under the restriction $p_1 = p2 = ... = p$.

Chapter 2

Billiard representation

Investigation of stochastic properties of the gravitational and scalar fields can be carried out straightforwardly in the framework of the generalized BLK model. However there is an elegant approach suggested, to our knowledge, first by Chitre [8] (see also Ref. [38]) for the case of mixmaster (Bianchi-IX) Universe and taken up recently in Refs. [24, 26, 29] in connection with description of statistical properties of inhomogeneities and multidimensional generalizations. This approach reduces the BLK model to a geodesic flow (billiard) on a manifold of a constant negative curvature. The properties of such geodesic flows are well studied [1]. There are theorems proving ergodicity and mixing for geodesic flows on compact negative curvature manifolds [32]. The simplest way to understand of why such stochastic properties arise is to consider the deviation n^i of neighboring geodesics which obeys the Jacobi equation

$$\frac{D^2 n^i}{ds^2} + R^i_{klm} u^k u^l n^m = 0 ,$$

where $u^k = dx^k/ds$ is the unit tangent vector of a geodesic line and in the case of a constant negative curvature $K < 0$ this gives

$$\frac{d^2}{ds^2} n^i + K n^i = 0 .$$

This equation has solutions in the form

$$n^i = n^i_1 e^{-\sqrt{-K}s} + n^i_2 e^{\sqrt{-K}s}$$

which demonstrate explicitly strong instability of the corresponding geodesic flow as $s \to \pm\infty$. If the phase space is compact the deviations cannot grow infinitely and this results in the strong mixing of geodesic trajectories. In what follows we shall not discuss these properties in more details and just use the above expressions to estimate the rate of growth of inhomogeneities.

1. Misner-Chitre approach

The main feature of the action for BLK model is that it does not contain space derivatives and is split up in the sum of independent local actions describing the gravitational and scalar fields at a particular point of the coordinate manifold S . It is convenient to re-write the action (3.29) in harmonic variables. Then the action (3.29) takes the form formally coinciding with the action for a continuous set of relativistic particles

$$I = \int_S \left(P_r \frac{\partial z^r}{\partial t} - \lambda' (P_i^2 + V_{\infty} - P_0^2) \right) d^n x dt, \tag{1.1}$$

where $r = 0, ..., n$, $i = 1, ..., n$, $\lambda' = \frac{\lambda}{n(n-1)}$, and new harmonic variables are related to the old ones with the linear transformation

$$q^a = A_j^a z^j + z^0 \ , \ z^n = \sqrt{\frac{2}{n(n-1)}} \phi$$

where $j = 1, ..., n-1$, and the constant matrix A_j^a obeys the conditions

$$\sum_a A_j^a = 0, \qquad \sum_a A_j^a A_k^a = n(n-1)\delta_{jk} \ , \tag{1.2}$$

and can be expressed in the form

$$A_j^a = \sqrt{\frac{n(n-1)}{j(j+1)}} (\theta_j^a - j\delta_j^a), \qquad \theta_j^a = \begin{cases} 1, & j > a \ , \\ 0, & j \le a \ . \end{cases}$$

For the sake of convenience we present also the explicit form of this matrix

$$A_j^a = \sqrt{n(n-1)} \begin{pmatrix} \frac{1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{2\cdot 3}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ -1 \cdot \frac{1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{2\cdot 3}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ 0 & -2 \cdot \frac{1}{\sqrt{2\cdot 3}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & -(n-1)\frac{1}{\sqrt{n(n-1)}} \end{pmatrix}$$

Since the timelike variable z^0 varies during the evolution as $z^0 \sim \ln g$, the positions of the potential walls turn out to be moving. It is more convenient to fix the positions of the walls. This can be done by using the so-called Misner-Chitre-like variables [24, 26] ($\vec{y} = y^j$)

$$z^0 = -e^{-\tau} \frac{1+y^2}{1-y^2}, \qquad \vec{z} = -2e^{-\tau} \frac{\vec{y}}{1-y^2}, \qquad y = \mid \vec{y} \mid < 1. \tag{1.3}$$

Using these variables, one can find the following expressions for the anisotropy parameters:

$$Q_a(y) = \frac{1}{n} \left\{ 1 + \frac{2A_j^a y^j}{1+y^2} \right\}, \tag{1.4}$$

which are now independent of the timelike variable τ.

We note that in the vacuum case the expressions (1.4) give, under the restriction $|y| = 1$, a parametrization of the standard Kasner exponents [5]. From (1.4) one can find the range of the anisotropy functions $-\frac{n-2}{n} \leq Q_a \leq 1$.

Choosing the quantity τ as a time variable, that is, using the gauge

$$N = \frac{n(n-1)}{2}\sqrt{g}\exp(-2\tau)/P^0]$$

we put the action (1.1) into the ADM form

$$I = \int_S \left\{ \vec{P}\frac{\partial}{\partial\tau}\vec{y} + P^n\frac{\partial}{\partial\tau}x^n - P^0(P,y) \right\} d^nxd\tau, \tag{1.5}$$

where the quantity

$$P^0(P,y) = \left\{ \epsilon^2(\vec{y},\vec{P}) + V[y] + (P^n)^2e^{-2\tau} \right\}^{1/2} \tag{1.6}$$

plays the role of the ADM Hamiltonian density and

$$\epsilon^2 = \frac{1}{4}(1-y^2)^2\vec{P}^2. \tag{1.7}$$

The part of the configuration space connected with the variables $\vec{y}$ is a realization of the $(n-1)$-dimensional Lobachevsky space [1] and the potential V cuts a part of it. Thus, locally (at a particular point of S) the action (1.6) describes a billiard on the Lobachevsky space. The positions of the walls which form the boundary of the billiard are determined, as a result of (3.25), by the inequalities (see also [6, 10])

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, \qquad a \neq b \neq c\ , \tag{1.8}$$

and the total number of the walls is $\frac{n!}{2!(n-3)!}$. Using the matrix (1.2) one can find that the walls are formed by spheres determined by the equations

$$\begin{aligned} \sigma_{abc} &= \tfrac{n-1}{n(1+y^2)}\left\{ (\vec{y} + \vec{B}_{abc})^2 + 1 - B^2_{abc} \right\} = 0, \\ \vec{B}_{abc} &= \tfrac{1}{n-1}(\vec{A}^a - \vec{A}^b - \vec{A}^c), \end{aligned} \tag{1.9}$$

where for arbitrary a, b, c we have $B^2 = 1 + \frac{2n}{n-1}$.

2. Properties of the Billiards

In the general case n points of the billiard having the coordinates

$$\vec{P}_a = \frac{1}{n-1}\,\overrightarrow{A_a}\ , \quad \overrightarrow{A_a} = \left(A_1^a, \ldots A_{n-1}^a\right)$$

lie on the absolute (on infinity of the Lobachevsky space). The trajectories which end at these points correspond to the set of Kasner exponents (0,...,0,1). When $n = 9$ there appear additional isolated points S_{abc} lying on the absolute. The coordinates of these points are given by the vectors

$$\vec{S}_{abc} = \frac{1}{12}(\ \overrightarrow{A_a} +\ \overrightarrow{A_b} +\ \overrightarrow{A_c})\ ,\quad a \neq b \neq c\,.$$

Despite the fact that these points lies on infinity the volume at these points is finite. To prove this fact let us consider the Poincare coordinates of the Lobachevsky space. Such coordinates can be introduced as follows

$$\overrightarrow{\eta} = 2\frac{\overrightarrow{y} + \overrightarrow{b}}{(y+b)^2} - \overrightarrow{b}\ , \tag{2.10}$$

where the vector $\overrightarrow{b}$ define a point on the absolute ($b^2 = 1$). In terms of these coordinates the metric of the Lobachevsky space takes the form

$$dl^2 = \frac{4\left(d\overrightarrow{y}\right)^2}{\left(1-y^2\right)^2} = \frac{\left(d\overrightarrow{\eta}\right)^2}{(b\eta)^2}\,.$$

These coordinates give realization of the Lobachevsky space in the form of a half-space $(b\eta) \geq 0$. Indeed, they transform the unit disc $1 - y^2 \geq 0$ into the half-space $1 - y^2 = \frac{4(\eta b)}{(\eta+b)^2} \geq 0$. The absolute of the Lobachevsky space is now the hyper-plane $(b\eta) = \eta_b = 0$ and infinity of the half-space $\eta_b \to \infty$. Besides, the transformation (2.10) remove the point $\overrightarrow{y} = -\overrightarrow{b}$ into infinity $\eta_b \to \infty$. Now it can straightforwardly be checked that in the case when the billiard has one point on the absolute the volume of the billiard is finite. Indeed, removing this point on infinity by the transformation (2.10) we find that the volume at this point is determined by the integral

$$V \sim const \int^{\infty} \frac{d\eta_b}{(\eta_b)^{n-1}}$$

which in the case of $n > 2$ is regular at the upper limit.

In the case of $n \geq 10$ in addition to the points P_a and S_{abc} there appear open accessible domains on the absolute [10, 29] and the volume of the billiard becomes infinite. If, on the contrary, $n < 10$, the volume of the billiard is finite and the billiard turns out to be a mixing one. In order to illustrate the billiards we give two simplest examples in Fig. 1. The case $n = 3$ in Fig. 1a coincides with the well-known mixmaster model and in Fig. 1b we illustrate the case

$n = 4$ which was first considered in Ref. [6]. In Fig. 1c we illustrate also the billiard $n = 3$ in the Poincare coordinates (2.10).

Now we show that, indeed, the billiards in the dimensions exceeding $n = 9$ become infinite. Let us introduce the new set of variables connected with the old ones as

$$\vec{x} = \frac{2\vec{y}}{1+y^2} . \tag{2.11}$$

Within these variables the absolute of the Lobachevsky space keeps the old position $|x|^2 = 1$ and the walls become planes [see (1.4), (1.9)]. We note also that in terms of these variables the trajectories of the billiard become ordinary straight lines. Furthermore, it will be more convenient to select a region on the Lobachevsky space in which the anisotropy parameters are in the increasing order

$$Q_0 \leq Q_1 \leq ... \leq Q_{n-2} \leq Q_{n-1}$$

and which is restricted by only one wall [see (1.8)] $\sigma(\vec{x}) = \sigma_{0,n-2,n-1}$. This region is formed by the vectors of the type

$$\vec{x} = \sum_{i=1}^{n-1} u^i \vec{e}_i ,$$

where the parameters $0 \leq u^i \leq 1$ and the set of basic vectors is given by

$$\begin{aligned} \vec{e}_i &= \tfrac{1}{n+i} \textstyle\sum_{a=i}^{n-1} \vec{A}^a \quad , i < n-2 \\ \vec{e}_{n-2} &= \tfrac{1}{2(n-1)} (\vec{A}^{n-2} + \vec{A}^{n-1}), \\ \vec{e}_{n-1} &= \tfrac{1}{n-1} \vec{A}^{n-1} \end{aligned} \tag{2.12}$$

They are normalised so that

$$\sigma(\vec{e}_i) = 0 \quad .$$

It is easy to find that the wall causes restrictions on the parameters u^i:

$$\sum u^i \leq 1 \quad .$$

The Euclidean norms of the basic vectors are

$$e_i^2 = \tfrac{i(n-i)(n-1)}{(n+i)^2} \quad , \; i < n-2,$$

$$e_{n-2}^2 = \tfrac{n-2}{2(n-1)} \quad , \; |e_{n-1}| = 1$$

here we used the following property of the vectors $\vec{A}^a$:

$$\sum_{k=1}^{n-1} A_k^a A_k^b = n(n-1)\delta^{ab} - (n-1) \, .$$

Now, it is easy to find that for $n < 9$ all basic vectors except $\vec{e}_{n-1}$ have norms less than unity and we have $|\vec{x}| \leq 1$ (the equality is achieved only when $\vec{x} = \vec{e}_{n-1}$).

In the case $n = 9$ we get $e_3^2 = e_8^2 = 1$. All other vectors have norms less than unity and we have the similar situation as above (i.e., $|x| = 1$ only when $\vec{x} = \vec{e}_3$ and $\vec{x} = \vec{e}_8$).

In the case of $n > 9$ a number of basic vectors have norms exceeding unity, e.g., $\vec{e}_i$ for $i = [\frac{n}{3}]$ or $i = [\frac{n}{3}] + 1$, where $[\frac{n}{3}]$ denotes the entire part of the number $\frac{n}{3}$. This means that the wall in these directions lies outside the absolute of the Lobachevsky space and there appears an open accessible domain. In other words, in these directions trajectories do not meet any obstacle and run to infinity. This proves the statement made above.

3. Dynamics of inhomogeneities

The system (1.5) has the form of a direct product of homogeneous local systems. Each local system in (1.5) has two functions $\epsilon(x)$ and $P^n(x)$ as integrals of motion. The solution of this local system for the remaining functions represents a geodesic flow on a manifold with a negative curvature. As is well known, a geodesic flow on a manifold with negative curvature is characterized by exponential instability [1]. This means that during the motion along a geodesic the normal deviations grow no slower than the exponential of the traversed path ($\xi \simeq \xi_0 e^s$), where the traversed path is determined by the expression

$$s = \int_{\tau_0}^{\tau} dl = \int_{\tau_0}^{\tau} \frac{2\,|\,\frac{\partial y}{\partial \tau}\,|}{(1-y^2)} d\tau = \frac{1}{2} \ln \left| \frac{P^0 - \epsilon}{P^0 + \epsilon} \right|_{\tau_0}^{\tau} . \tag{3.13}$$

This instability leads to a stochastic nature of the corresponding geodesic flow. The system possesses the mixing property [32] and an invariant measure induced by the Liouviulle one:

$$d\mu(y, P) = const \times \delta(E - \epsilon) d^{n-1} y d^{n-1} P, \tag{3.14}$$

where E is a constant. Integrating this expression over ϵ, we find

$$d\mu(y, m) = const \times \frac{d^{n-1} y d^{n-2} m}{(1 - y^2)^{n-1}}, \tag{3.15}$$

where $\vec{m} = \frac{\vec{P}}{\epsilon}$, $|m| = 1$.

Since the inhomogeneous system (1.5) is a direct product of homogeneous systems, one can simply describe its behavior. In particular, the scale of inhomogeneity decreases as

$$\lambda_i \sim \left(\frac{\partial y}{\partial x}\right)^{-1} \sim \lambda_i^0 \exp(-s)\,, \tag{3.16}$$

and after sufficiently large time ($s(\tau) \to \infty$) the dynamical functions $\vec{y}(x)$, $\vec{P}(x)$ become random functions of the spatial coordinates. In order to calculate different mean values one can use the n-point distribution functions [24]

$$\rho_{x_1,\ldots,x_n}(y_1,\ldots,y_n,m_1,\ldots,m_n) = \left\langle \prod_{i=1}^{n} \delta(y_i - y(x_i))\delta(m_i - m(x_i)) \right\rangle , \tag{3.17}$$

where the angular brackets can denote averaging either over an initial distribution or over a certain coordinate volume $\Delta V \gg (\lambda_i^0)^3$. The mixing results in the relaxation of initial functions (3.17) to the limiting ones which have the form of the direct product of measures (3.15): $d\mu = \prod_i d\mu_i$. Thus, the asymptotic expressions for averages and correlating functions take the form

$$\begin{gathered} \langle \vec{y} \rangle = \left\langle \vec{P} \right\rangle = 0 \ , \\ \langle y_k(x), y_l(x') \rangle = \langle y_k, y_l \rangle \, \delta(x, x'), \end{gathered} \tag{3.18}$$

where $|x - x'| \gg \lambda_i^0 \exp(-s)$.

Here it is necessary to point out the role of the scalar field in the dynamics and statistical properties of inhomogeneities. As may be easily seen from (3.13), in the absence of the scalar field (i.e., $P^n = 0$) the traversed path coincides with the duration of motion [we have $s = \Delta\tau = \tau - \tau_0$ instead of (3.13)]. Thus the effect of scalar fields is displayed in the replacement of the dependence for the traversed path on the time variable and, therefore, in the replacement of the rate of increase of the inhomogeneities. This replacement does not change qualitatively the evolution of the Universe in the case of cosmological expansion. But in the case of a contracting Universe the situation changes drastically. Indeed, in the limit $\tau \to -\infty$ from (3.13) we find that the traversed path s takes a limited value s_0 and therefore the increase of the inhomogeneities turns out to be finite. One of the consequences of such behavior is the fact that at the singularity the functions $\vec{y}$ and $\vec{P}$ take constant values. In other words, in the presence of scalar fields the cosmological collapse ends with a stable Kasner-like regime (2.17). This fact may be seen in another way. Indeed, in the limit $\tau \to -\infty$ the scalar field gives the leading contribution to the ADM Hamiltonian (1.6) and P^0 does not depend on the gravitational variables.

The finiteness of the traversed path $s(\tau)$ leads, generally speaking, to the destruction of the mixing properties [32], since for the establishment of the invariant measure it is necessary to satisfy the condition $s_0 \to \infty$. Evidently, this condition requires the smallness of the energy density for the scalar field as compared with the ADM energy of the gravitational field [the last term in (1.6) in comparison with the first ones]. Indeed, in this case s_0 is determined by the expression $s_0 = -\ln \frac{P^n e^{\tau_0}}{2\varepsilon}$, which follows from (3.13), and as $P^n \to 0$ one gets $s_0 \to \infty$ (i.e., s can have arbitrarily large values).

Thus, in the case of cosmological contraction one may speak of mixing and, therefore, of the establishment of an invariant statistical distribution only for those spatial domains which have a sufficiently small energy density of the scalar field.

4. Estimates and Origin of cellular structure of space

In this manner the large-scale structure of space in the vicinity of a singularity acquires a quasi-isotropic nature. The distribution of inhomogeneities is determined by the set of functions of the spatial coordinates $\epsilon(x)$, $\Pi_\phi(x)$, and also R^c_α and T^α_a which conserve, during the evolution, a primordial degree of inhomogeneity of the space. The scale of inhomogeneity for the other functions decreases as $\lambda \approx \lambda_0 e^{-s(\tau)}$. In this section we give some estimates clarifying the behavior of the inhomogeneities. For simplicity we consider the case when the scalar field is absent.

To find an estimate for the growth of inhomogeneities in synchronous time t $(dt = N d\tau)$ we set $\vec{y} = 0$. Then, for the variation of the variable τ one may find the estimate

$$\sqrt{g} \sim \exp(-\frac{n}{2} e^{-\tau}) \sim P^0 t$$

where the point $t = 0$ corresponds to the singularity. According to (3.16) the dependence of the coordinate scale of inhomogeneity on the time t takes the form

$$\lambda \approx \lambda_0 \ln(1/g_0)/\ln(1/g)$$

in the case of a contracting Universe $(g \to 0)$ and

$$\lambda \approx \lambda_0 \ln(1/g)/\ln(1/g_0)$$

in the case of an expanding Universe.

The rapid generation of smaller and smaller scales leads to the formation of spatial chaos in the metric functions and the large-scale structure of the Universe acquires a quasi-isotropic nature. Speeds of the scale growing (Hubble constants) for different directions turn out to be equal after averaging out over spatial domains having a size $\approx \lambda_0$. Indeed, using (1.4), (3.15) one may find expressions for the averages $< Q_a >= 1/n$.

In addition, it is necessary to mention one more characteristic feature of the oscillatory regime in the inhomogeneous case. This is the formation of a cellular structure in the scale functions Q_a during the evolution which demonstrate explicitly the stochastic process of the development of inhomogeneities [21]. Indeed, let us consider some region of coordinate space ΔV. The functions $\vec{y}$ define a map of that region on a domain $\Sigma \in K$. During the evolution the size of the domain Σ grows $\simeq e^{s(\tau)}$ and Σ covers the billiard K many times. Each covering determines its own preimage in ΔV. In this manner the initial coordinate volume turns out to be split up into "cells" $\Delta V = \bigcup_i \Delta V_i$. In every cell ΔV_i the vector $\vec{y}$ takes almost all admissible values $\vec{y} \in K$ and that of the functions Q_a ($Q_a \in [Q_{min}, 1]$ where $Q_{min} = -\frac{(n-1)^2-(n+1)}{n(n+1)}$). To illustrate this process let us consider the case $n = 3$. In this case it is convenient to use the Poincarè model of the Lobachevsky plane on the upper complex half-plane $H = \{W = U + iV, V \geq 0\}$ [see fig. 1(c)]. The line $V = 0$ is called the absolute and its points lie at infinity. Geodesics in H are given by semicircles with centers on the absolute or by rays which are perpendicular to the absolute.

The billiard is the region $K \in H$, bounded by the geodesic triangle $\partial K = [|W| = 1, U = \pm 1]$. The area of the billiard is equal to π. The motion can be continued to the whole plane H. For this aim one needs to reflect the domain of the billiard with respect to one of the boundary walls and make the iteration of such procedure. In this way the Lobachevsky plane will be covered by a set of domains K^n , each of which is connected with the region of the billiard K by a one-to-one mapping. During the evolution an arbitrary initial square Σ^0 begins to grow and covers more and more domains K^n [see fig.1(c)].

The cellular structure pointed out above turns out to depend on the time and the number of the cells increases as $N \approx N_0 e^{s(\tau)}$. However, the situation is changed if we consider a contracting space filled with a scalar field. Then the evolution of this structure in the limit $g \to 0$ ends, because the functions Q_a become independent of time, and on the final stage of the collapse one would have a real cellular structure [21].

It is also necessary to note that from the point of view of the $n-$dimensional volume ΔV every cell is topologically a torus or a cylinder. However in the multidimensional case we should take into account the fact that the physically observable space has the number of dimensions equal to three. Then from the point of view of a $3-$dimensional volume every cell turns out to be bounded in every direction, i.e., it is topologically a sphere.

In spite of the isotropic nature of the spatial distribution of the field the large local anisotropy displays itself in an anomalous dependence of spatial lengths for vectors and curves upon the time variable. Indeed, a moment of the scale function $< g^{MQ_a} >$ (where $M > 0$) decreases when $g \to 0$ as the Laplace integral

$$< g^{MQ_a} > = \int_{Q_{min}}^{1} g^{MQ_a} \rho(Q_a) dQ_a \ ,$$

where $\rho(Q_a)$ is the distribution which follows from (3.15). The main contribution to this integral is given by the point $Q = Q_{min}$ .In the case $n > 3$ in the limit $(Q - Q_{min}) \to 0$ one can find $\rho(Q) \approx C(Q - Q_{min})^{n-2}$, where C is a constant and we obtain the estimate

$$\left\langle g^{MQ_a} \right\rangle \approx \frac{g^{MQ_{min}}}{(M \ln 1/g)^{n-1}}. \tag{4.19}$$

This expression shows that for $n > 3$ average lengths even increase while approaching the singularity. The case $n = 3$ must be considered separately. In this case we have $Q_{min} = 0$ and the explicit form of the distribution function $\rho(Q_a)$, as follows from (3.15), is

$$\rho(Q) = \frac{2}{\pi}(Q(1-Q))^{-1/2}(1+3Q)^{-1}.$$

As $Q \ll 1$ one has $\rho(Q_a) \approx \frac{2}{\pi}(Q_a)^{-1/2}$ and, thus, in the limit $g \to 0$ we get the estimate [21]

$$\left\langle g^{MQ_a} \right\rangle \approx (M \ln(1/g))^{-1/2}.$$

So in the case $n = 3$ the averaged scales are decreasing when $g \to 0$ but only logarithmically.

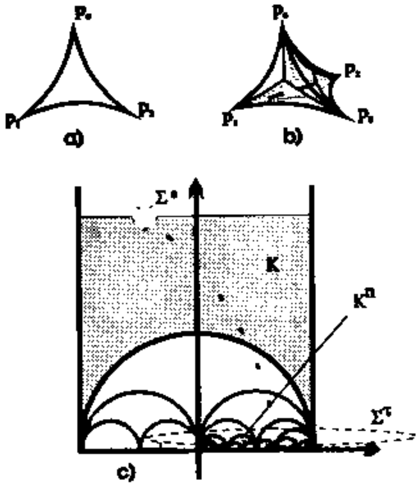


Fig. 2.1 : (a), (b) The regions of the billiards in the cases $n = 3$ and $n = 4$, respectively. The points Pi lie on the absolute $|y|^2 = 1$ of the Lobachevsky spaces. (c) The upper complex half-plane represents the two-dimensional Lobachevsky plane. The painted region K represents the billiard. K_n is a set of images of K. $\sum^\circ$ is an arbitrary initial square. $\sum^\tau$ is the same square after the time τ.

Chapter 3

Quantum cosmology

1. introduction

In this lecture we considered the behavior and properties of an inhomogeneous gravitational field in the vicinity of the singularity from the quantum point of view. We shall use the models constructed in the previous lectures. These models are the generalized Kasner (GK) model and the BLK models describing in the case of $n < 10$ the general oscillatory behavior of the metric near the cosmological singularity.

Beforehand it is necessary to recall why these models turn out to be suitable in quantum description of the singularity. As is well known there are two different types of the evolution of the Universe [33]. The first one is the normal evolution when the Universe decelerates $\ddot{a} < 0$,i.e., $a \sim t^{\alpha}$ with $\alpha < 1$, (where a is the scale factor of the Universe and for the sake of simplicity we suppose the Universe to be isotropic one). The second type is the inflationary evolution when the Universe accelerates $\ddot{a} > 0$ ($\alpha > 1$). We present a possible alternation of these types of the evolution of the Universe on Fig.2.

It is important that in the very beginning the expansion of the Universe is always described by the first type of the evolution only (the normal evolution). Therefore, inhomogeneities of the metric acquire large-scale character $l_i \gg l_h$ and can be described in the framework of the GK or BLK models. Indeed, as was shown earlier the inhomogeneities behave as $l_i \sim a$ and in the case when the Universe decelerates $l_i \sim t^{\alpha} \gg l_h \sim t$ as $t \longrightarrow 0$ (with $\alpha < 1$).

It is also necessary to mention the fact that when quantizing those models we meet the main difficulty of quantum gravity , that is, the non-renormalizability. To overcome this difficulty we consider simply short distance fluctuations to be omitted. In other words we shall assume implicitly the existence of a sufficiently small cut off parameter in Fourier transforms of fields and, therefore, the space coordinates will take a discrete values. This will allow us to regularise all expressions which appear in inhomogeneous cases.

Another important fact is that the Kasner-like parametrization of dynamical variables divides these variables into two groups. The first group contains scale functions of Kasner vectors and their momenta and behaves near the singularity like a set of ordinary coupled scalar fields (at least from the point of view of their contribution in the dynamics). In conventional gravity ($n = 3$) just this group represents, with the Hamiltonian constraint being taken into account, the true gravitational degrees of freedom, for the momentum constraint relates tightly the residual variables with the scale functions. In the presence of additional dimensions the second group contains residual variables only a part of which turns out to be non-dynamical. However this group has behavior like a set of vector fields and, therefore, in leading order it gives no or only an implicit contribution to the dynamics of the metric (in particular, just the second group variables provides the existence of potential walls in the BLK models).

Thus , when quantizing the inhomogeneous multidimensional models we do not take into account the presence of the second group variables as we do for the ordinary matter having an equation of state $\varepsilon > p$. In other words we shall consider only scale functions to be quantum degrees of freedom and do not consider the rest of degrees of freedom at all.

2. local dynamics of inhomogeneous models

2.1. Quantization. The Wheeler-De Witt equation.

As it was shown the action for GK and BLK models (1.1) takes the form formally coincided with the action for a continuous set of relativistic particles (in the case of the GK model $V_\infty = 0$)

$$I = \int_S \left\{ P_r \frac{\partial z^r}{\partial t} - \lambda' (P_i^2 + V_\infty (z) - P_0^2) \right\} d^n x dt, \tag{2.1}$$

where $r = 0, ..., n$, $i = 1, ..., n$, $\lambda' = \frac{N}{n(n-1)\sqrt{\beta}}$.

The configuration space M of the system (2.1) (called also superspace) is represented in the form of the direct product $M = \prod_{x \in S} M_x$. Moreover, every local space M_x is the ordinary $n+1$-dimensional pseudo-Euclidean space, for the kinetic term, that determines a metric on M_x, turns out to be coincided with that of for the ordinary flat $n+1$-dimensional pseudo-Euclidean spacetime manifold. The fact that M_x has the pseudo-Euclidean structure is connected with that the set of scale functions z^r contains one odd field variable z^0 which reflects the freedom in the choice of a cosmological time. Just this variable has timelike character and can, in quantum cosmology, serve as an inner time variable.

Since, as was mentioned above the action (2.1) resembles the action for a continuous set of relativistic particles, quantisation of such a system may be carried out in the complete analogy with that of relativistic particles [42]. The zero-energy Hamiltonian constraint leads to the set of the Wheeler-DeWitt equations [12]

$$(-\Delta_x + U_x + \xi P_x)\Psi = 0, \qquad x \in S, \tag{2.2}$$

where Ψ is the wave function of the universe, Δ_x denotes a Laplace operator on M_x : $\Delta_x = \frac{1}{\sqrt{-G}}\partial_A\sqrt{-G}G^{AB}\partial_B$, G_{AB} is the metric on M_x determined by the interval

$$\delta\Gamma(x)^2 = \frac{1}{4\lambda'}((\delta x^i(x))^2 - (\delta x^0(x))^2), \tag{2.3}$$

P_x is the curvature scalar of M_x. The value of ξ should be chosen as $\xi = \frac{n-1}{4n}$ to provide a conformal invariance of Eq.(2.2) which reflects the arbitrariness in the choice of the lapse function λ . Indeed, the transformation

$$G_{AB} \to \tilde{G}_{AB} = e^{-2\Omega}G_{AB}, \quad \Psi \to \tilde{\Psi} = e^{\frac{n-1}{2}\Omega}\Psi$$

maps the Eq.(2.2) into

$$(-\tilde{\Delta}_x + e^{2\Omega}U_x + \frac{n-1}{4n}\tilde{P}_x)\tilde{\Psi} = 0.$$

and the theory becomes independent on a particular choice of λ.

To solve the equation (2.2) we shall consider a lattice approximation. To this end we shall suppose the existence of a sufficiently small minimal scale of inhomogeneity for all fields l_{min} , so that the coordinates x will take discrete values only. The continuous limit one obtains tending l_{min} to zero, though, from the other side, one may think of the lattice model as of a background model and treat the scales less than l_{min} as small perturbations.

The system of equations (2.2) turns out to be uncoupled, for each from these equations contains a set of functions which are specified at a particular point x of S . In the classical case this constitutes the condition pointed out that the inhomogeneities has only large scales $l_i \gg l_h$. In quantum theory such inequality must be fulfilled for mean values $\langle l_i\rangle \gg \langle l_h\rangle$ and, therefore, this implies an appropriate restriction on quantum states of the fields. In what follows we shall call the sets of degrees of freedom M_x at a different points of S as $x-$sets. Therefore, the space H of solutions to this system takes the form of the tensor product of spaces H_x ($H = \prod_{x\in S} H_x$) as that of M, where H_x is the space of solutions to a particular $x-$ equation (2.2). Accordingly, all $x-$sets of degrees of freedom may independently be considered. Therefore, at first it will be convenient to work out the probability interpretation and all the technique on the example of a one local $x-$set of degrees of freedom and after that to generalize the consideration to the case of all degrees of freedom.

2.2. The space of solutions to the WDW equation for a particular x-set of degrees of freedom.

Every local x-equation (2.2) admits the conserved current

$$J_A(\Psi,\Psi) = i[\Psi^*\nabla_A\Psi - \Psi\nabla_A\Psi^*] \tag{2.4}$$

which may be used to determine the inner product in the spaces H_x

$$<\varphi \mid \chi> = i\int_{\Sigma_x} J_A(\varphi,\chi)d\Sigma_x^A, \tag{2.5}$$

where ∇_A denotes a covariant derivative on x-metric (2.3), Σ_x is an arbitrary space-like surface on M_x , and $d\Sigma_x^A = d\Sigma_x n^A$, with $d\Sigma_x$ being the volume element in Σ_x, and n^A being the timelike unit vector normal to Σ_x. The main property of this inner product is the independence of the choice of hypersurface Σ_x .

In the case of generalized Kasner model the potential V is absent and the complete set of solutions to the WDW equation (2.2) can be found exactly. Those are the well known modes of free particles moving in $n+1$ - dimensional spacetime

$$u_p = (2\pi)^{-n/2}\frac{1}{\sqrt{2p_0}}e^{-ipx} \ , \tag{2.6}$$

with $p_0 = |p|$.

In the case of BLK model to obtain solutions in an explicit form turns out to be impossible. Nevertheless, such solutions can be constructed formally. Indeed, let us consider the Misner-Chitre like variables ($\vec{y} = y^j$, $j = 1,...,n-1$)

$$z^0 = -e^{-\tau}\frac{1+y^2}{1-y^2}, \qquad \vec{z} = -2e^{-\tau}\frac{\vec{y}}{1-y^2}, \qquad y = \mid \vec{y} \mid \leq 1 .$$

In terms of these variables the potential $V(z)$ becomes independent of the timelike variable τ and the supermetric takes the form

$$\delta\Gamma(z)^2 = \frac{e^{-2\tau}}{4\lambda'}\left(\frac{4(\delta y^j)^2}{(1-y^2)^2} + e^{2\tau}(\delta z^n)^2 - (\delta\tau)^2\right). \tag{2.7}$$

For the sake of simplicity in what follows we shall use the gage $4\lambda' e^{2\tau} = 1$.

As was pointed out earlier the part of the configuration space M_x related to the variables $\vec{y}$ is a realization of the $(n-1)$-dimensional Lobachevsky space and the potential V_∞ cuts a part K of it

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, \qquad a \neq b \neq c$$

which in the case $n \leq 9$ has a finite volume. Let us suppose that there is a set of solutions to the eigenvalue problem for the Laplace - Beltramy operator

$$(\Delta_y + k_J^2 + \frac{(n-2)^2}{4})\varphi_J(x) = 0, \qquad \varphi_J\,|_{\partial K} = 0, \tag{2.8}$$

where the Laplace operator Δ_y is constructed via the metric $dl^2 = h_{ij}dy^i dy^j = \frac{4(dy)^2}{(1-y^2)^2}$ and J collects all indexes numbering the eigenfunctions φ_J. In the case of $n < 10$ the region K has a finite volume and J takes discrete values $(J = 0, 1, 2, \ldots)$, while for $n \geq 10$ the volume of K is infinite and the spectrum of the Laplace - Beltramy operator becomes continuous one. Unfortunately this function cannot be obtained in an explicit form but , nevertheless one can find them numerically with any degree of accuracy. The eigenfunctions φ_j obey the orthogonality and normalization relations

$$(\varphi_I, \varphi_J) = \int_K \varphi_I^*(y)\varphi_J(y)d\mu(y) = \delta_{IJ}, \tag{2.9}$$

where

$$d\mu(y) = \frac{1}{c}\sqrt{h}d^{n-1}y = \frac{2^{n-1}}{c}\frac{d^{n-1}y}{(1-y^2)^{n-1}},$$

and c is the volume of K . The completeness conditions for this functions has the form

$$\sum_I \varphi_I^*(y)\varphi_I(y') = \frac{\delta(y-y')}{\sqrt{h}}.$$

Then a complete orthonormal set $\{u_p, u_p^*\}$ of solutions to $x-$equation (2.2) constitute functions of the form

$$\begin{aligned} u_p &= \exp(-\tfrac{1}{2}\tau)\chi_p(\tau)\Phi_p(y,z) \ , \\ \Phi_p(y,z) &= (2\pi)^{-1/2}\varphi_J(y)\exp(i\epsilon z^n) \end{aligned} \tag{2.10}$$

where $p = (J, \epsilon)$. Functions $\chi_p(\tau)$ satisfy the equation following from (2.2):

$$\frac{d^2\chi_p}{d\tau^2} + \omega_p^2(\tau)\chi_p = 0, \qquad \omega_p^2(\tau) = k_J^2 + \epsilon^2 e^{-2\tau} \tag{2.11}$$

with the normalization condition

$$\chi_p^*\frac{d\chi_p}{d\tau} - \chi_p\frac{d\chi_p^*}{d\tau} = -i \ , \tag{2.12}$$

and are expressed via Bessel functions. The initial conditions to Eq.(2.11) at a moment τ_0 are to be taken in the form

$$\chi_p(\tau_0) = \frac{1}{\sqrt{2\omega_p(\tau_0)}}, \quad \chi_p'(\tau_0) = -i\omega_p(\tau_0)\chi_p(\tau_0), \tag{2.13}$$

which imply that functions χ_p are positive frequency functions at the moment τ_0 .

The set of solutions (2.10) is orthonormal in the sense of the scalar product (2.5), i.e., they satisfy the relations

$$< u_p \mid u_q >= - < u_p^* \mid u_q^* >= \delta_{pq}, \qquad < u_p \mid u_q^* >= 0. \tag{2.14}$$

Thus, an arbitrary solution f to the local Wheeler-DeWitt equation (2.2) can be represented in the form

$$f = \sum_p A_p^+ u_p + A_p^- u_p^*, \tag{2.15}$$

where $A_p^\pm$ are arbitrary constants which are to be specified by initial conditions.

2.3. The Hilbert space and the probability interpretation

Since the norm determined by the scalar product (2.5) turns out to be sign-indefinite we face up with the difficulty of the probability interpretation. The simplest way to define a positive-definite inner product is to separate a submanifold H_x^+ on the space H_x which is of "positive frequency". If we suppose $A_p^- = 0$ in (2.15), then the normalisation condition for f takes the form

$$f^+ = \sum A_p^+ u_p \ , \ < f \mid f >= \sum_p \mid A_p^+ \mid^2 = 1, \tag{2.16}$$

and meets no difficulties. Indeed, H_x^+ is a linear space and the inner product has all necessary properties required for a scalar product

$$\begin{gathered} < \chi \mid \varphi >=< \chi \mid \varphi >^* , \\ < \chi_1 + \chi_2 \mid \varphi >=< \chi_1 \mid \varphi > + < \chi_2 \mid \varphi > , \\ < \chi \mid \chi > \geq 0 , \end{gathered} \tag{2.17}$$

the equality in (2.17) is reached only when $\chi = 0$. Thus, the subspace of physical states H_x^+ becomes the ordinary Hilbert space and we can adopt the standard probability interpretation [42].

In the absence of the scalar field (i.e. when $\epsilon = 0$) the positive frequency modes (2.11) takes the form

$$\chi_p = \frac{1}{\sqrt{2k_J}} e^{-ik_J\tau} , \tag{2.18}$$

and the restriction on physically admissible states (2.16) is equivalent to the use of the ADM procedure for quantization of the gravitational field [2]. In other words, the same theory will be obtained if we use for quantisation the action functional in the form (1.5):

$$I = \int \left\{ \vec{P} \frac{\partial}{\partial \tau} \vec{y} - P_0 (y, P) \right\} d\tau,$$

with $P_0(y,P) = \sqrt{\epsilon^2(\vec{y}, \vec{P}) + V[y]}$ being the ADM Hamiltonian density. Indeed, in this case the wave function (2.16) obeys the first-order wave equation

$$i\partial_\tau \mid f^+ >= P_0 \mid f^+ > . \tag{2.19}$$

and the states (2.14), (2.18) turn out to be eigenstates for the operator P_0

$$P_0 u_J = k_J u_J \tag{2.20}$$

and define the stationary states of the gravitational field. We stress, however, that the geometry described by these states is nonstationary, for the metric functions contain the timelike variable τ in an explicit form.

Now we recall one important fact which is known from the particle physics and concerns of physical observables. The fact is that the zeroth component of the super-current (2.4) $J^0(z)$ cannot be interpreted as the probability density to find the field at a point z^a of a hypersurface Σ^0_z of the configuration manifold M_z , for despite the fact the inner product is positively defined for states (2.16), nevertheless, the current density is not. In order to construct the probability density we have to use the construction suggested by Newton and Wigner (see Ref. [42]). In the case when the scalar field is absent the state describing the field localised at the moment τ at the point $y^i \in \Sigma^\tau_z \subset M_z$ is given by the expression

$$\Phi (y, \tau) = \sum_J \sqrt{k_J} u^*_J (y, \tau) u_J . \tag{2.21}$$

We note that these states can be found by solving the eigenvalue problem for the operator of the field coordinate $\widehat{z}$. In the case of Kasner model positive frequency solutions of the WDW equations are the ordinary plane waves (2.6) $u_p = (2\pi)^{-n/2} \frac{1}{\sqrt{2p_0}} e^{-ipz}$ and the expression (2.21) yields

$$\Phi_{y,0} \left(z^i, 0 \right) = \int d^n p \sqrt{p_0} u^*_p (y) u_p (z) = const \cdot \left(\frac{1}{|z - y|} \right)^{n - \frac{1}{2}}$$

from which one can see that the localised states do not coincide with the Dirac's delta-function. Accordingly, if the field is described by a wave function $\Psi = \sum_J A^+_J u_J$ then the probability for the scale functions to be localized at the point y is given by

$$P (y, \tau) = |(\Phi (y, \tau) | \Psi)|^2 .$$

Thus, for an arbitrary chosen initial state we get the probability density in the form

$$P (y, \tau) = \left| \sum_J \sqrt{k_J} u^*_J (y, \tau) A^+_J \right|^2 .$$

2.4. The ambiguity in the choice of the Hilbert space

It is clear that the procedure of the choice of the Hilbert space (H_x^+ in the H_x) is not uniquely defined.

In a general case the Wheeler-DeWitt equation contains the timelike variable in an explicit form and, therefore, it is impossible to classify solutions by means of signs of frequencies. The solutions (2.10) being of "positive frequency" at a given moment of time τ_0 (2.13) are a mixture of both frequencies at an arbitrary moment. This happens, in particular, when we include in the consideration a matter (e.g., in the presence of the scalar field).

Thus, instead of the set of modes (2.10) one can use another complete set of modes $\left\{v_p, v_p^*\right\}$ connected with (2.10) by a Bogoliubov transformation

$$v_p = \sum_p \left\{\alpha_{pq} u_q + \beta_{pq} u_q^*\right\} \ , \tag{2.22}$$

where the coefficients α_{pq} and β_{pq} satisfy the relations

$$\begin{aligned} \Sigma_q \left\{\alpha_{pq}\beta_{p'q}^* - \alpha_{pq}^*\beta_{p'q}\right\} = \delta_{pp'} \ , \\ \Sigma_q \left\{\alpha_{pq}\beta_{p'q} - \alpha_{p'q}\beta_{pq}\right\} = 0 \ , \end{aligned} \tag{2.23}$$

which follows from the condition that the new set (2.22) is orthonormal. Making use the new set (2.22) we can determine a new Hilbert space $\widetilde{H_x^+}$, which constitutes vectors of the type

$$\widetilde{f^+} = \sum_p C_p v_p. \tag{2.24}$$

It is clear that the spaces H_x^+ and $\widetilde{H_x^+}$ are coincided in the case when all coefficients β_{pq} equal to zero. However, if amongst of β_{pq} there are some different from zero then H_x^+ and $\widetilde{H_x^+}$ do not coincide and we will have thus two unitary nonequivalent ways to construct a quantum theory.

We note that similar ambiguity is also contained in the ADM approach, since the ADM Hamiltonian can be chosen by different ways (one can choose an arbitrary timelike variable as a time and take its conjugate momentum as a Hamiltonian). We also note that this ambiguity is of principal an inherent difficulty of quantum cosmology, which apparently can be only solved in the framework of third quantization. In particular, the following fact shows that the third quantization is necessary. As well known in quantum gravity there is a principal restriction of the observability of physical fields (e.g., see [11]), and this restriction has complete analogy with the well known restriction on the observabilty of coordinates of relativistic particles.

3. The presence of matter and the case of all degrees of freedom

3.1. The presence of the scalar field. Mixing of frequencies and third quantization.

As was pointed out above in the presence of a scalar field the Wheeler-DeWitt equation turns out to be explicitly time-dependent, see, Eq. (2.11). In this case there is no a unique way to determine positive frequency solutions (the mixing of frequencies is said to occur). In this connection the procedure of choice of the Hilbert space H_ε^+ is also ambiguous. In the analogy with the relativistic particle theory one can attempt to solve this problem in a framework of the second ("third") quantization [41, 19, 36, 23]. In this section we shall consider a homogeneous case, i.e., the field variables z to be independent on space coordinates. For the sake of simplicity we put the spatial volume $V(S) = \int_S d^n x$ to be equal to unity.

Third quantization represents also independent interest, from the point of view of two possible applications. Firstly, as an example of a theory allowing to describe some simple topology changes (in this case we mean the changes of the number of disconnected copies of S). And secondly, it is a theory giving a possibility to describe the process of "quantum creation of the World from nothing" proposed in [13] (see [23]). Here, we will discuss the second possibility only.

In this case we have just one Wheeler-DeWitt equation (2.2)

$$(-\Delta + U + \xi P)\Psi = 0\ , \tag{3.25}$$

which can be obtained from a variation principal if one writes the action functional for the wave function in the form

$$S = \frac{1}{2}\int (G^{AB}\partial_A\Psi^*\partial_B\Psi - (U + \xi P)\mid\Psi\mid^2)\sqrt{-G}d^{n+1}z\ . \tag{3.26}$$

A complete orthonormal set $\left\{u_p, u_p^*\right\}$ of solutions of eq. (3.25) constitute the functions (2.10), (2.11). In order to account for the mixing of frequencies in an explicit form, it is convenient to represent the function $\chi_p(\tau)$ in form

$$\chi_p = (2\omega_p)^{-1/2}[\alpha_p(\tau)e^{-i\Theta_p} + \beta_p(\tau)e^{i\Theta_p}] \tag{3.27}$$

$$\partial_\tau\chi_p = i(\omega_p/2)^{1/2}[\alpha_p(\tau)e^{-i\Theta_p} - \beta_p(\tau)e^{i\Theta_p}], \tag{3.28}$$

where

$$\Theta_p(\tau) = \int_{\tau_0}^{\tau}\omega_p d\tau, \tag{3.29}$$

and $\alpha_p(\tau)$ and $\beta_p(\tau)$ are complex functions to be determined which obey the initial conditions (2.13)

$$\alpha_p(\tau_0) = 1, \beta_p(\tau_0) = 0, \tag{3.30}$$

which denotes that the modes u_p are of positive-frequency in the moment $\tau = \tau_0$. The representation of $\partial_\tau \chi_p$ in the form (3.28) removes the arbitrariness in the determination of two functions α and β via the single function χ. The condition (2.12) provides the validity of the equality

$$\mid \alpha_p \mid^2 - \mid \beta_p \mid^2 = 1 \tag{3.31}$$

for an arbitrary τ.

When a third quantization is imposed the wave function of the universe becomes a field operator and can be expanded in the complete orthonormal set (2.10) of solutions of Eq.(3.25) (for the sake of simplicity we assume below that Ψ is real):

$$\psi = \sum_p A_p u_p + A^+ u_p^* \ , \tag{3.32}$$

where the operators A_p and A_p^+ obey the following commutation relations

$$[A_p, A_{p'}^+] = \delta_{p,p'}. \tag{3.33}$$

The Hamilton operator for quantum field is determined by the equality

$$H(\tau) = \int_{\tau=const} \theta_{00}(\Psi) G^{00} \sqrt{-G} d^n z \tag{3.34}$$

where θ_{AB} is the "superenergy-momentum" tensor specified on M (on the configuration manifold) which can be obtained by variation of action (3.26) on metric G^{AB}.

Return-Path: ¡kirillov@vxrmg9.icra.it¿ Received: from vxrmg9 (vxrmg9.icra.it) by lafexSu1 (4.1/SMI-4.1) id AA06425; Thu, 26 Oct 95 21:43:13 EST Message-Id: ¡9510270043.AA06425@lafexSu1¿ Date: Thu, 26 Oct 1995 22:01:12 +0200 From: kirillov@vxrmg9.icra.it To: bacg@lafexSu1.lafex.cbpf.br Subject: Part 2 text of lectures of A.Kirill X-Vms-To: SMTPStatus: R

Since the basis functions (2.10) obey the initial conditions (3.30), i.e., are of positive-frequency in the moment $\tau = \tau_0$, then the decomposition (3.32) determines a particle interpretation at this moment. In particular, at this moment of time, a vacuum state determined by the equalities

$$A_p \mid 0 >= 0 \tag{3.35}$$

for arbitrary p, coincides with the ground state of the Hamiltonian (3.34). In an arbitrary moment of time $\tau \neq \tau_0$ the modes (2.10) contain both the positive frequency part and the negative frequency part as well (3.27) and the vacuum determined by (3.35) does not coincide already with the ground state of the Hamiltonian (3.34). In other words, the state $\mid 0 >$ for $\tau \neq \tau_0$ describes already an excited state of the quantum field Ψ. In order to determine the ground state of the Hamiltonian (3.34) in an arbitrary moment of time we introduce the field operators $b_p(\tau)$ and $b_p^+(\tau)$ depending of time [15]:

$$\begin{aligned} b_p(\tau) &= \alpha_p(\tau)A_p + \beta_p^*(\tau)A_p^+ \ , \\ b_p^+(\tau) &= \alpha_p^*(\tau)A_p^+ + \beta_p(\tau)A_p \ , \end{aligned} \tag{3.36}$$

where the functions α_p and β_p are given in (3.27), (3.28). The conditions (3.34) lead to the following commutation relations of the operators $b_p(\tau)$ and $b_p^+(\tau)$:

$$[b_p(\tau), b_{p'}^+(\tau)] = \delta_{pp'}. \tag{3.37}$$

The Hamiltonian (3.34) being expressed via these operators takes the diagonal form

$$H(\tau) = \frac{1}{2}\sum_p \omega_p(\tau)[b_p(\tau)b_p^+(\tau) + b_p^+(\tau)b_p(\tau)] \ , \tag{3.38}$$

and its ground state determined by the relations

$$b_p(\tau) \mid 0, \tau >= 0 \tag{3.39}$$

turns out to be depending of time variable. It is obvious that in the moment $\tau = \tau_0$ the ground state (3.39) coincide with the vacuum state that is determined by the relations (3.35), i.e., $\mid 0 >=\mid 0, \tau_0 >$. In any arbitrary moment the state $\mid 0 >$ differs from (3.39). Then in each p-mode it contains

$$N_p(\tau) =< 0 \mid b_p^+(\tau)b_p(\tau) \mid 0 >=\mid \beta_p(\tau) \mid^2 \tag{3.40}$$

field quanta.

In this manner the operators $b_p(\tau)$ and $b_p^+(\tau)$ define the depending of time particle interpretation [15]. Then (3.40) may be interpreted as the process of quantum creation of field quanta (universes) from "nothing" [41, 23].

Let the field Ψ be in the ground state in the asymptotic region of M_s corresponding to a singularity. Then the initial conditions (3.30) are specified at the moment $\tau_0 = -\infty$. Solutions (2.11) satisfying the initial conditions (3.30) takes the form

$$\chi_p(\tau) = \frac{1}{2}(\pi)^{1/2}\exp(\pi k_J/2)H_{ik_J}(R), \tag{3.41}$$

where $H_{ik_J}(R)$ is the Hankel function, and $R = \epsilon e^{-\tau}$. The dependence on time of functions α_p and β_p is determined by (3.27), (3.28), while the dependence upon time for the spectral number density of universes produced is determined by the expression

$$N_p(\tau) = |\beta_p(\tau)|^2 = (2\omega_p)^{-1}(|\partial\chi_p/\partial\tau|^2 + \omega_p^2 |\chi_p|^2) - 1/2, \tag{3.42}$$

where $\chi_p(\tau)$ is given in (3.41).

The expressions (3.42) can be significantly simplified in the case when the modes satisfy the inequality:

$$\epsilon \ll k_J e^{\tau}. \tag{3.43}$$

Then the functions α_p and β_p tend to constant values

$$\begin{aligned} \alpha_p &= \{\exp(\pi k_J)/2sh(\pi k_J)\}^{1/2} e^{i\theta_p^+}, \\ \beta_p &= \{\exp(-\pi k_J)/2sh(\pi k_J)\}^{1/2} e^{i\theta_p^-}, \end{aligned} \tag{3.44}$$

where $\theta_p^{\pm}$ are constant phases, and (3.42) takes the form

$$N_p = (\exp(2\pi k_J) - 1)^{-1}, \tag{3.45}$$

which formally coincides with the Planckian distribution corresponding to the temperature $T_0 = \frac{1}{2\pi}$.

Since the number of universes is a variable quantity, a description of a single universe cannot be given by a pure state[1]. States of the universe will be described by a density matrix. Making use of (3.42) we can find that the density matrix describing a Universe created from "nothing" takes form

$$\rho = N^{-1}(\tau) \sum_p | u_p > N_p < u_p |, \tag{3.46}$$

where N is a normalization constant which coincides with the total number of created universes (which can be estimated as $N(\tau) \approx N_0 e^{\tau}$), and

$$| u_p >= (4\pi\omega_p)^{-1/2} \exp(\tau/2) e^{i\Theta_p} \varphi_J(y) \exp(i\epsilon x^n). \tag{3.47}$$

Here we note that the positive frequency functions (3.47), generally speaking, satisfy the Wheeler-DeWitt equation only under inequalities (3.43).

[1] In the framework of the theory describing a single universe this makes impossible to introduce the probability interpretation [46].

3.2. The case of all degrees of freedom

Now the generalization to the case of all degrees of freedom may be carried out straightforwardly. The positive frequency sector H^+ in the total space of solutions H we determine as the direct product of positive frequency local submanifolds $H^+ = \prod_{x \in S} H_x^+$. Thus, the wave function takes the form

$$\Psi = \sum_{[p(x)]} F_{p(x)} U_{p(x)}, \qquad U_{p(x)} = \prod_{x \in S} u_{p(x)} \tag{3.48}$$

with the scalar product induced by (2.14)

$$\langle \chi | \psi \rangle = \sum_{[p(x)]} B^*_{p(x)} A_{p(x)} \ , \tag{3.49}$$

where $\chi = \sum B_{p(x)} U_{p(x)}$ and $\psi = \sum A_{p(x)} U_{p(x)}$ are arbitrary vectors lying in H^+ .

Despite that Eq. (3.48) and (3.49) give a well defined probability interpretation in the inhomogeneous case, it is necessary to mention that the procedure of the choice of H^+ in the H acquires an additional ambiguity. Indeed, now the Bogoliubov transformation (2.22) relating different sets of modes of local manifolds H_x can contain dependence on spatial coordinates $x \in S$

$$v_{p,x} = \sum_q \left\{ \alpha(x)_{pq} u_q + \beta(x)_{pq} u_q^* \right\} \tag{3.50}$$

which will determine a new total Hilbert space $\widetilde{H} = \prod_{x \in S} \widetilde{H_x^+}$ and what is still worse we are already unable to relate basises in these spaces by a Bogolubov transformation of the type (3.50). Therefore, the probability interpretation and all relevant constructions turn out to be crucially dependent upon the particular choice of the physical sector H^+ in H .

3.3. The BLK inhomogeneous model, stationary states and statistical properties of metric inhomogeneities

Now we consider properties of inhomogeneities of the metric in the case of The BLK model. For the sake of simplicity we shall consider the case when the scalar field is absent and the number of spatial dimensions less than ten [in this case the configuration space is bounded in terms of the anisotropy parameters Q]. In this case to obtain solutions in an explicit form turns out to be impossible. States of the field will be represented by a set of positive-frequency modes for the Wheeler-DeWitt equation, and will have the structure of the direct product of modes for homogeneous field : $| \, [J(x)] >= \prod_{x \in S} u_{J(x)}$. In the case of $n \leq 9$ these modes are classified by an integer-valued function $J(x)$. As well as in the case of a particular H_s these states are the eigenstates of the ADM Hamiltonian density

$$P_0(x)\,|[J(x)]\rangle = k_{J(x)}\,|[J(x)]\rangle \tag{3.51}$$

and they may be considered as stationary states. The ground state of the system $|0\rangle$ (the vacuum state from the point of view of field excitations) is given by the direct product of eigenfunctions corresponding to the minimal value k_0 and has a bounded energy density $k_0 < \infty$.

Let us consider now the properties of inhomogeneities of the field. The basic variables of the theory which characterize the inhomogeneities gravitation field are the operator functions $z(x)$ and their momenta $p(x)$. A state of the field is described by either a wave function $\mid \Psi >$ or a density matrix $\hat{\rho}$ (this depends upon that pure or mixed state the system is given in). The space distribution of inhomogeneities of the field can be described by different mean values $< z(x) >, < p(x) >$ of the operators and corresponding correlation functions of $< z^i(x), z^k(x') >$ type. The brackets $< \cdot >$ denote the average over the state of the field.

In contrast to the classical theory, expressions for these quantities are essentially dependent upon the choice of the initial conditions (upon the choice of an initial quantum state). It turns out that the properties of inhomogeneities are "most classical" in the stationary states $|[J(x)]\rangle$ (or in the states described by a stationary density matrix). It is easy to obtain the expressions

$$\begin{aligned} &< z(x) > = < p(x) > = 0\,, \\ &< z^i(x), z^k(x') > = (z^i z^j)_{J(x)}\, \delta(x,y)\,, \end{aligned} \tag{3.52}$$

for these states, where $(z^i z^k)_J = \sum_m (z^i)_{Jm} \left(z^k\right)_{mJ}$, and z_{pq} is a matrix element for the local space H_x^+ ($z^i_{pq} = < u_p \mid z^i \mid u_q >$). The delta-function appears in (2.2) due to the fact that the wave function $|[J(x)]\rangle$ has the structure of the direct product (3.48) and we use obvious properties of stationary (localised in terms of z) states $z_{JJ} = p_{JJ} = 0$.

¿From (3.52) one can see that the intensity of fluctuations depends upon the space coordinates. This dependence disappears only for $J(x) = J = const$ (e.g., for the vacuum state $J = 0$). Obviously, expression (3.52) should converges, in the asymptotic $J \to \infty$, to the formulas obtained in the classical case (3.18) and the coordinate dependence should disappear.

Nonstationary states display more essential differences from the classical averages. For example, consider the state $\mid f >= \int_S f(x) \mid 1_x > d^n x$ which describes a unique excitation distributed in the space with the density $\mid f(x) \mid^2$. Then using (3.51) one obtains

$$< f \mid z(x) \mid f > = < f \mid p(x) \mid f > = 0, \tag{3.53}$$

$$< f \mid z^i(x_1) z^k(x_2) \mid f > = \begin{cases} \left(z^i z^k\right)_{11} \mid f(x) \mid^2 + \left(z^i z^k\right)_{00} (1 - \mid f(x) \mid^2)\,, & x_1 = x_2 \\ f^*(x_1) f(x_2) \left(z^i z^k\right)_{01} + f^*(x_2) f(x_1) \left(z^i z^k\right)_{10}\,, & x_1 \neq x_2 \end{cases}$$

instead of (3.52).

3.4. Estimates and dynamical reduction of extra dimensions in Inhomogeneous Kaluza-Klein cosmological models

In this manner properties of inhomogeneous models depend crucially upon the choice of initial data. Despite this when $n \leq 9$, near the singularity the behavior of lengths in time shows universal features. For mean scale factor we get $< a_i >=< g^{Q_i} >\sim cg^{Q_{\min}}$ as $g \to 0$, where g is the metric determinant which near the singularity may serve as a time variable, $Q_{\min} = -\frac{n-3}{n+1}$ is the minimal admissible value of the anisotropy parameters Q_i and c is a slowly varying with g function, collecting quantum corrections, and differing from the classical one. The fact that the exponent $Q_{\min}$ is negative does not mean that the quantum theory avoids the singularity. Those quantities describes the behavior of lengths in the vicinity of the singularity and is a result of the strong local anisotropy. It follows simply from that the main contribution in averages gives just those regions of the configuration space on which Q_i is negative. When we are interested in averages of the anisotropy parameters (the Hubble constants for different directions) we shall find that they are almost always positive, e.g. for stationary states we have $< Q_i >= \frac{1}{n}$ like in the classical theory, however in the general case the state being arbitrary chosen gives $< Q_i >$ to be an oscillating function of g . The more adequate quantity may be the volume element for an arbitrary chosen hypersurface Ξ^m . In particular for $m = n$ we get $< V >= \sqrt{g} \to 0$ as $g \to 0$, therefore, from the n-dimensional point of view, the cosmological collapse is inevitable.

To investigate the behavior of $m-$squares we, at first, recall important formulas related to the billiards. As in the previous section we shall consider vacuum case (below we follow Refs. [30, 31]).

The positive frequency solutions to the WDW equations have the structure (2.10)

$$u_J = \frac{1}{\sqrt{2k_J}} \exp(-ik_J\tau)\varphi_J(y) \ , \tag{3.54}$$

with φ_J being solutions to the eigenvalue problem (2.8)

$$(\Delta_y + k_J^2 + \frac{(n-2)^2}{4})\varphi_J(z) = 0, \qquad \varphi_J \mid_{\partial K} = 0 \ . \tag{3.55}$$

In the case of $n < 10$ the billiard K has a finite volume and J takes discrete values ($J = 0, 1, 2, \ldots$), while for $n \geq 10$ the volume of K is infinite and the spectrum of the Laplace - Beltramy operator becomes continuous one. Accordingly, in the first case the eigenfunctions φ_J are localised in terms of the anisotropy parameters Q . An arbitrary wave function Ψ describing the gravitational field at the point x is represented in the form (2.16)

$$\Psi = \sum_J A_J u_J \ , \ < \Psi \mid \Psi >= \sum \mid A_J \mid^2 = 1,$$

where A_J are arbitrary constants which are to be specified by initial conditions. We recall that the states (3.54) play the role of stationary states (though they describe non-stationary geometries). The probability for the scale functions to be localised at the point $y \in M_x$ is given by

$$P(y,\tau) = |\langle \Phi(y,\tau) | \Psi \rangle|^2$$

with $\Phi(y,\tau) = \sum_J \sqrt{k_J} u_J^*(y,\tau) u_J$ being the Newton-Wigner localised states (2.21). Thus, for an arbitrary chosen initial state we get

$$P(y,\tau) = \left| \sum_J \sqrt{k_J} u_J^*(y,\tau) A_J \right|^2 = \left| \sum_J \frac{1}{\sqrt{2}} \exp(ik_J\tau) \varphi_J^*(y) A_J \right|^2 .$$

Now let us consider an arbitrary m-dimensional hypersurface $\Xi^m \subset S$. The volume element for this hypersurface has the form

$$dV^m = \sum g^{\mu_{a_1,\dots,a_m}} C_{a_1,\dots,a_m} \ell^{a_1} \bigwedge \dots \bigwedge \ell^{a_m},$$

where

$$\mu_{a_1,\dots,a_m} = \frac{1}{2} \sum_{i=1}^{m} Q_{a_i}$$

and $C_{a_1,\dots,a_m}$ are arbitrary constant functions of spatial coordinates. Thus, the behavior of this element in time is determined by the quantities g^{μ_m} . In quantum theory such quantities are operators and have to be averaged out.

Near the singularity ($g \to 0$) when $n \leq 9$ the main contribution in the mean value $\langle g^{\mu_m} \rangle$ is given by a small neighborhood of just that point of the spacelike part K of the configuration space M at which the exponent μ_m takes the minimal value. Such point lies on the boundary of ∂K and the values of such exponents is given by

$$\mu_m^* = \mu\left(\overrightarrow{e_m}\right) = -\frac{m(n-m-2)}{2(n+m)}$$

for $m < n-2$ and $\mu_{n-2}^* = \mu_{n-1}^* = 0$ and $\mu_n \equiv \frac{1}{2}$ where $\overrightarrow{e_m}$ are the basic vectors (2.12). In particular, $2\mu_1$ gives the minimal possible value of the anisotropy parameters $Q_{\min}$. Since $\varphi_J(\partial K) = 0$, in the neighborhood of ∂K we have $\varphi_J \approx \eta_J(\mu - \mu^*)$ and the probability density is (we suppose $n > 3$)

$$P_\tau(\mu) = \int_K P(y,\tau)\, \delta(\mu - \mu(y)) \sqrt{h} d^{n-1}y \approx B_m(\tau) (\mu - \mu^*)^n$$

as $\mu \to \mu^*$. Thus, we find that in the limit $g \to 0$ moments of the function g^{μ_m} are given by $(L > 0)$

$$\left\langle (g^{\mu_m})^L \right\rangle = D_m(L,\tau) \frac{\left(g_*^{\mu_m^*}\right)^L}{(L \ln 1/g_*)^{n+1}} , \tag{3.56}$$

where $g_* = g(\tau, y^*)$ and D_m is slowly varying in time function which collects information of the initial quantum state. Since $\mu_m^* < 0$ for $m < n-2$, one can see that an arbitrary hypersurface Ξ^m whose dimension less then $n-2$ expands while approaching the singularity. In this sense one can consider the cosmological collapse to be the mechanism reducing the number of spatial dimensions down to $n-3$. On the contrary, in the early expanding Universe an arbitrary hypersurface Ξ^m, whose dimension less than $n-2$, contracts. We stress that this does not solve yet the problem of reduction of additional dimensions and shows just an initial tendency to such reduction. The problem of its subsequent stage and its stability remains still open [9]. In this manner, the number of dimensions of the early Universe can be effectively reduced down to three. The law of expansion of the three-dimensional space which is conjugated to an arbitrary $n-3$-hypersurface can be estimated as $V_3 \sim g^{\frac{3}{2}k} \sim t^{3k}$ with $k = \frac{2}{3}\left(\frac{1}{2} - \mu_{n-3}\right) = \frac{n-2}{2n-3}$ and it corresponds to an effective equation of state $p = \frac{n}{3n-6}\epsilon$.

When considering dimensions exceeding $n = 9$ the situation changes drastically. The potential in (3.55) does not restrict the spacelike part of the configuration space and, therefore, we have no localized, in terms of the exponents μ, states. If we get ready a localized state (a wave packet) the width of the packet spreads eventually more and more out and, simultaneously, the center of the wave packet runs to infinity of the configuration space. In classical theory this signals us that the oscillatory mode becomes unstable and transforms into a Kasner-like behavior. Therefore, different mean values will depend upon the initial state crucially.

We also recall that in dimensions less or equal to nine the evolution of the metric undergoes spontaneous stochastization which was shown to be described by an invariant measure. Using this measure we can also evaluate the behavior of average $m-$volumes. Though the estimates turn out to be the same as in (3.56) [with the replacement $n \to n-2$ and already constant D_m] such estimates has more restricted sense, since the need to introduce a probability distribution is here the result of an uncertainty in initial data. In quantum theory, however, the description is probabilistic from the very beginning and only averages have physical sense.

In conclusion we briefly repeat the main features of the quantum consideration of the inhomogeneous cosmological models.

1. It is always possible to construct a quantum inhomogeneous cosmology in a self-consistent manner. We mean here that it is possible to construct the Hilbert space and define a probability interpretation (despite the fact that the choice of the Hilbert space is ambiguous).

2.Near the singularity in the absence of scalar fields one can determine stationary states of the metric (which describe non-stationary geometries). These states are classified by filling numbers $n(x)$ (the density of excitations which, in the three dimensional case, correspond to the density of frozen gravitons whose wavelengths exceed the horizon sides).

3. Close to the singularity we have no a classical background. If we try to distinguish a classical background in the inhomogeneous models we shall find that all quantum corrections

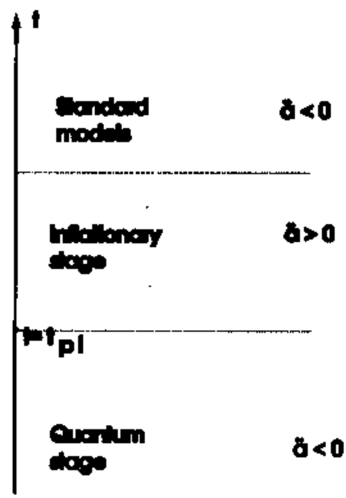
deverge near the singularity. Indeed, at every space point the interval can be represented in the form

$$ds^2 = -N(y,\tau)\,d\tau^2 + \sum g^{Q_a(y)}\,(dx^a)^2 \ .$$

In a leading order we find the estimates: $Q_a = \frac{1}{n} + f_a(y)$, $g \sim t^2$, where for stationary states we get $\langle f_a(y)\rangle = 0$. Then the expansion of this metric gives us the expression

$$ds^2 = -dt^2 + t^{\frac{2}{3}}\,(dx)^2 + t^{\frac{2}{3}} \sum \left(2f_a \ln t + (2f_a \ln t)^2 + \ldots\right)(dx^a)^2$$

from which one can easiely see that all corrections are divergent in the limit $t \longrightarrow 0$. Therefore, it is not correct to use a homogeneous and isotropic model as a starting point to quantum cosmology.



$ds^2 = dt^2 - a^2 dx^2$

Chapter 4

Third quantization and topology fluctuations in the early universe

1. Introduction

As is widely accepted quantum fluctuations of the metric at small scales can change the spacetime topology [49]. At the present Universe these fluctuations appears to be unobservable in direct experiments, for they occur at such small scales (of the Plankian size order) which are far too distant from those ones which are achieved in modern experiments. Nevertheless, in the very early Universe topology-fluctuation-effects might play an important role in formation of the overall structure of our Universe. Indeed, in the expanding Universe the scales, being small at the very beginning, become eventually large and acquire a cosmological significance. Therefore, in solving problems of realistic cosmological scenarios topology fluctuations should be taken into account.

The simplest processes connected with topology changes (wormholes and baby universes) are known to be described in the framework of third quantisation [17, 36]. In particular, third quantization seems to be the natural tool for description of quantum creation of a universe from nothing [41, 19] which was suggested in Ref. [13]. Let us recall briefly the basic ideas of the third quantisation.

Let Σ be a three-dimensional space manifold (our large "Mother" Universe) and $\{\psi_A\}$ be a set of quantum states for gravitational and matter fields specified on Σ which form the Hilbert space H of the ordinary "flat-space" quantum theory. Let us assume that a number of small closed universes S_i ("babies") can branch off, or joint onto, Σ . To describe the processes of such a kind we have to construct a new Hilbert $W = H \otimes F_S$. Where F_S is the Fock space for baby universes which can be constructed as follows. Let $\{f_i\}$ be the set of states of a closed homogeneous universe S. Let us introduce a set of creation and annihilation operators $\left\{a_i, a_i^+\right\}$ satisfying the commutation relations

$$\left[a_i \, , \, a_j^\dagger\right] = \delta_{ij} \; .$$

Then an arbitrary vector $\Psi \in W$ can be represented in the form

$$\Psi = \sum_A C^A |\psi_A\rangle + \sum_{A,i} C_i^A a_i^+ |\psi_A\rangle + \sum_{A,i,j} C_{ij}^A a_i^+ a_j^+ |\psi_A\rangle + \dots ,$$

where $|\psi_A\rangle$ represents a set of vacuums for a_i : $a_i |\psi_A\rangle = 0$ for all A.

The only annoying thing here is the fact that the theory of such a kind admits just simplest topology changes while to describe topology fluctuations we should be able to account for all possible topologies of the Universe. This can be done in the framework of a new approach pointed out in Ref. [28] which generalises the third quantization. Such generalization is based on the following circumstance. The fact is that the quantum topology fluctuations occur at very small scales while in real experiments we have to use macroscopic classical devices. Therefore, we have to describe arbitrary topologies in terms of the basic "macroscopic" coordinate manifold which seems to have the simplest "flat" topology. The base of topology of the basic manifold S is formed by a countable set $\{V_i\}$ of open neighborhoods V_i so that the unification of V_i gives S

$$S = \bigcup_{i \in I} V_i \ .$$

An example of such base is represented by sets of the type $V_{y,r} = \{x \in S \ , \ |x - y| < r\}$ where y and r are rational numbers. Then the construction of the Hilbert space for a complite quantum theory can be carried out as follows.

Let V_i be a particular neighborhood of a point y and $\{g_{i,A}\}$ be a set of quantum states for matter and gravitational field specified on V_i . Let us define the set of creation and annihilation operators $a_{i,A}$ and $a_{i,A}^+$ with commutation relations being

$$\left[a_{i,A} \ , \ a_{j,B}^+\right] = \delta_{ij}\delta_{AB} \ .$$

Using this algebra we can define vacuum state $|0\rangle$

$$a_{i,A}|0\rangle = 0 \ , \quad for \ \ all \ \ i \ , \ A$$

and construct the Fock space F. The vacuum state $|0\rangle$ describes the situation when the physical spatial continuum is absent and, therefore, there are no matter, no gravity, no observables. The states of the type $|g_{i,A}\rangle = a_{i,A}^+|0\rangle$ describe one point y_i with its neighborhood V_i and with matter and gravitational fields given in the quantum state $g_{i,A}$. For these states we have got the real physical space (the neighborhood V_i) and observables. The states $|N_{i,A}\rangle = \frac{1}{\sqrt{N!}}\left(a_{i,A}^+\right)^N |0\rangle$ describe the situation when the physical continuum contains N identical sets V_i given in a quantum state $g_{i,A}$. In general vectors in F will. be represented in the form

$$\Psi = C^0 |0\rangle + \sum_{i,A} C^{i,A} a_{i,A}^+ |0\rangle + \sum_{i,j,AB} C^{A,i,B,j} a_{i,A}^+ a_{j,B}^+ |0\rangle + \dots ,$$

where $|C^0|^2$ gives the probability of the absence of the physical space, $\left|C^{i,A}\right|^2$ gives the probability of that we have just one neighborhood V_i, etc.. It is clear that using sufficiently small V_i we can glue arbitrary topologies of the physical space. Besides, we obtain here the possibility to solve the problem of non-renormalizability of quantum gravity. Indeed, in the framework of the approach suggested one can construct the physical space having an arbitrary density $N(x)$. Thus the physical space can contain hollows at small scales, that is, $N(\mathbf{k}) \to 0$ if $\mathbf{k} \to \infty$ (where $N(\mathbf{k}) = (2\pi)^{-3/2} \int N(x) \exp(-i\mathbf{k}x) d^3x$) and this can be used to regularise divergent expressions in conventional quantum gravity.

We also point out that in quantum gravity the topology appears as a new dynamical characteristic of the physical space and, therefore, it is not fixed but is to be determined by dynamics. In this lecture we show how one can evaluate the effects of topology fluctuations in the evolving Universe. Since the most promising models of the early Universe are given in the framework of chaotic inflationary scenarios [45] we consider a quasi-homogeneous De Sitter model to describe quantum topology fluctuations. We note that despite the fact that inflationary models are able to provide, in general features, explanation of the observable picture of the Universe (homogeneity, isotropy, flatness, etc.) it was pointed out [43] that on sufficiently large distances exceeding the visible part of the present Universe one could expect the Universe to be essentially inhomogeneous and anisotropic[44, 35]. We shall show that it will not be so if quantum topology fluctuations are taken into account. More precisely, we show that during the evolution topology fluctuations increase and if in the very beginning the Universe had a rather simple topology the spacetime foam will almost completely determine properties of the universe.

2. The Hamiltonian formulation

Let S be a three-dimensional space manifold, which we shall assume to be compact $\partial S = 0$, and $g_{\alpha\beta}$, ϕ^j $(j = 1, ..., m)$ be metric functions and a set of scalar fields specified on S . So the 4-interval takes the form

$$ds^2 = N^2 dt^2 - g_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt) \tag{2.1}$$

and the action may be represented as follows [24] (we use Plankian units $m_p^2 = 16\pi$)

$$I = \int_S (p_A \frac{\partial}{\partial t} q^A + P_A \frac{\partial}{\partial t} y^A + \pi_j \frac{\partial}{\partial t} \phi^j - NC - N^\alpha C_\alpha) d^3x dt, \tag{2.2}$$

where $(A = 0, 1, 2,)$

$$C = \frac{1}{\sqrt{g}} (\sum_A p_A^2 - \frac{1}{2} (\sum_A p_A)^2 + \frac{1}{2} \pi^2 + W), \tag{2.3}$$

$$C_\alpha = p_A \partial_\alpha q^A + P_A \partial_\alpha y^A + \pi_j \partial_\alpha \phi^j, \qquad W = g(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi^j \partial_\beta \phi^j + V(\phi) -^3 R) \tag{2.4}$$

where we use the so-called Kasner-like parametrization [24, 29]

$$g_{\alpha\beta} = \sum_A e^{q^A} \ell^A_\alpha \ell^A_\beta, \quad \pi^\alpha_\beta = \sum_A p_A \ell^A_\alpha L^\beta_A, \quad \ell^A_\alpha = U_{AB}\partial_\alpha y^B, \tag{2.5}$$

and P_A is expressed via the matrix $U_{AB} \in SO(3)$ as $P_A = -2\partial_\alpha(\sum_{BC} p_B U_{BA} U_{BC} \partial x^\alpha / \partial y^C)$, (see, for more details, [24]). In what follows it will be more convenient to choose a harmonic set of the configuration variables

$$q^A = z^0 + D^A_i z^i, \quad \phi^j = \sqrt{3} z^{j+2}, \tag{2.6}$$

(here D^A_i is a constant matrix $D^A_i = \sqrt{\frac{6}{i(i+1)}}\left(\theta^A_i - i\delta^A_i\right)$, $\theta^A_i = \{^{1,\ i>A}_{0\ i\leq A}$, $i = 1, 2$) in which the Hamiltonian constraint (2.3) takes the form

$$C = \frac{1}{6\sqrt{g}}(p_k^2 - p_0^2 + 6W), \tag{2.7}$$

where p_a, ($a = 0, ..., n$, where $n = m + 2$) stands for the momenta conjugated to the harmonic variables z^a. Now one can consider y^A to be new coordinates on S and resolve the momentum constraints (2.4) with respect to P_A . Then we obtain a reduced action which can be read off

$$I = \int_S [p_a \frac{\partial}{\partial t} z^a - \lambda(p_k^2 - p_0^2 + 6W(z, p)] d^3x dt, \tag{2.8}$$

where $\lambda = \frac{N}{6\sqrt{g}} = \frac{N}{6}\exp(\frac{3}{2}z^0)$ and the potential W appears now as a function of all dynamical variables and momenta.

3. Generalized Kasner solution and quantization

Generalized Kasner solution is an automodel solution which realized under the condition

$$W(z, p) \ll T(p) \ ,$$

where T stands for the first two terms in (2.7). This implies the potential energy of gravitational and scalar fields becomes negligible as compared with the kinetic energy of the fields. Then from (2.8) we are ready to obtain the generalised Kasner solution [34]

$$ds^2 = N^2 dt^2 - \sum \exp\left(q^A\right) \ell^A_\alpha \ell^A_\beta dx^\alpha dx^\beta, \tag{3.9}$$

where $\ell^A_\alpha(x)$ are constant functions and the only evolving variables are the scale functions $q^A(x)$

$$z^a = z^a_0 + 2p^a \int_{t_0}^t \lambda dt \ ,$$

where $p^a = \gamma^{ab} p_b$, $\gamma_{ab} = diag(-1,+1,...,+1)$. In the gage $N = 1$ one gets $q^A = q_0^A + 2s^A \ln t$, where $s^A = (p^0 - D_i^A p^i)/p^0$ is the standard Kasner exponents [34] satisfying the identity $\sum s^A = \sum (s^A)^2 + (p^\perp/p^0)^2 = 1$ with $p_i^\perp$ being orthogonal to D_i^A, i.e. $D_i^A p_i^\perp \equiv 0$.

The configuration space of the system (2.8), the so-called superspace, can be regarded as the direct product of a continuous set of local $n+1$-dimensional Pseudo-Euclidean spaces $M = \prod_{x \in S} M_x$, where $n = m+2$. The kinetic term in (2.8) induces a metric on M which is determined by the superinterval

$$\delta\Gamma^2 = \int_S \delta\Gamma^2(x) d^3x = \int_S \frac{1}{4\lambda} \gamma_{ab} \delta z^a(x) \delta z^b(x) d^3x \ , \tag{3.10}$$

where $\delta z^a(x)$ can be regarded as a small change of dynamical functions z^a and λ is an arbitrary function.

Quantization is carried out by imposing the commutation relations

$$[z^a(x), p_b(y)] = i\delta_b^a \delta(x,y) \ .$$

This relations have the well known representation $p_A(x) = -i\nabla_A(x)$, where $\nabla_A(x)$ denotes the covariant derivative constructed on the supermetric (3.10). Then the Hamiltonian constraint (2.7) with the potential being neglected gives the set of Wheeler-DeWitt equations (3.12)

$$\Delta_x \Psi = \frac{1}{\sqrt{-G_x}} \partial_A \sqrt{-G_x} G_x^{AB} \partial_B \ \Psi = 0 \ , \quad x \in S \quad , \tag{3.11}$$

where G_x^{AB} is the local supermetric (the metric on M_x) given by the local interval $\delta\Gamma^2(x)$ (3.10). Because of the absence of the potential term the set of solutions to these equations may be obtained in the explicit form

$$f_p^\pm = A \exp\left(\pm i \int_S p_a z^a d^3x\right) \tag{3.12}$$

with functions $p_a(x)$ satisfying the equation $\gamma^{ab} p_a p_b = 0$ and A being a normalisation constant. The momenta p_a can be parametrised as $p_a = \omega n_a$, with an arbitrary function $\omega(x)$ and vector $n_a = (1,\ n_k(x))$, $n_k n^k = 1$.

Thus, it can be seen that we have almost complete analogy with relativistic particles. The set of variables z^a plays the role of spacetime coordinates and the label x numbers sorts of particles [42]. Therefore, we shall follow that analogy as far as it is possible. As is well known in the particle physics solutions of the wave equation (in our case of the WDW equation) are divided into two groups of positive and negative frequency. To be admissible from the physical point of view the wave function of a particle must contain only those modes which are of positive frequency. Therefore, one should try to make a similar division in the case of gravity. Setting $z^0(x) = const$ (i.e. synhronising time variables) we rewrite (3.12) as $f_p^\pm \sim e^{\pm iEz^0}$, where $E = \int_S \omega d^3x$ may be called the total ADM (Arnowitt-Deser-Misner) energy [2]. Here, however, one faces the first obstacle. Indeed, in gravity one supposes all of the configuration variables and their momenta to be differentiable functions of x . Therefore, one is unable to classify solutions with respect to the sign of frequencies, for $\omega(x)$ is an arbitrary function which can change the sing when one traces it over S. We can divide S into two submanifolds S^+ and S^- with respect to the sing of $\omega(x)$ and, therefore, in the general case the solutions f^+ (3.12) which one presumes to be of positive frequency have the structure $f^+(S) = f^+(S^+)f^-(S^-)$,i.e., they are really of positive frequency on S^+ and of negative frequency on S^-. And moreover, even the sign of the total ADM energy turns out to be indefinite. We cannot simply consider the function ω to be everywhere positive, otherwise we exclude essential part of states. Indeed, from the classical point of view the sign of ω shows whether the local volume of S expands or contracts and the both cases are admissible on the classical level. All this means the well known fact that z^0 is unable, in the general case, to be a true time variable. This difficulty may be overcome by one of the following two methods . Either by choosing a new time variable T in such a way that $|\partial_t T| = |\partial_t z^0|$ and the momentum conjugated to T is a function having a definite sing, or by adding a small mass term to the WDW equation (3.11) (something like a cosmological constant Λ but having different scalar weight). We note also that there are a number of cases of interest when z^0 can, nevertheless, serve as a true time variable, for values of p_0 turn out to be separated from zero by a slot. These are the case of an inflating universe [45, 27] and the case of closeness to the cosmological singularity [30]. In what follows we shall consider z^0 to be a good appropriately chosen timelike variable, at least for the background model. Thus, in the same manner as in particle physics we can determine the positive frequency solutions to be describing physical states of the generalised Kasner model (GKM).

The next difficulty is faced when we try to account for the potential term. Indeed, since GKM is just a model, one can hope to use it as a firs step of an approximation procedure to quantum gravity. In this manner we can consider a green function and expand amplitudes in rows by a small parameter. Thus, in the first order we get the diagram illustrated on Fig. 3a. If the potential (the perturbating term) is sufficiently small we do not meet here any difficulty at all. The main difficulty appears in the second order when we regard diagrams illustrated on Fig.3b,c. Fig.3b corresponds to the case when the scattering on the potential occurs in such a way that the intermediate states are of positive frequency, i.e., during the process the zeroth component of the intermediate momenta $p_0'(x)$ remains to be positive at every point $x \in S$. In the other words the processes of such a type occur without the frequency mixing.

However, in the second order we get also diagrams of the type illustrated on Fig. 3c. In the last case the scattering is accompanied with frequency mixing, that is illustrated by that the intermediate momentum p' is directed backward with respect to the time variable ($p_0'(x) < 0$). In actually, the frequency mixing may occur not on the whole basic manifold S but on a part of it $K \subset S$ and we have to integrate over all possible submanifolds K , even those ones which contain just a number of particular points. On the classical level this means that the variable being chosen as a time cannot serve as time any more. Having a particular trajectory we can redefine the time variable and, thereby, to improve the situation. In quantum theory, however, if we do so we just break the situation on Fig 3a and Fig. 3b. In the other words we just draw the trouble in another place. All this signals us that we are in principle unable to choose a good time variable to fit all possible trajectories. This is, as I think, the main reason of why we have not got a good definition of time in quantum gravity so far.

Thus, we have got in some way to interpret the negative frequency solutions (3.12). Of course, for rather simple potential terms or in the case of a linear theory we can merely neglect the "negative energy" solutions or even solve the problem exactly without meeting any difficulties. But in the general case it turns out to be impossible. We note that it is not a new problem for quantum theory, for we had met such situation in the particle physics. Exactly as in QFT the frequency mixing signals us that we have a "particle production" and the problem becomes a multiparticle one. In quantum gravity this points out to topology changes (fluctuations) discussed first by Wheeler [49] and, more recently, in connection with wormholes and baby universes in Ref. [17]. That, in particular, clarifies us why in quantum gravity one is unable to measure field variables with an arbitrary degree of accuracy (e.g. see [11]). Indeed, as we are just going to localize field functions at a particular domain of the configuration space (of the superspace) we, thereby, create simultaneously additional pieces of the spacetime manifold (in the other words, we change topology) and all "one-particle" observables lose sense. The smaller volume of the configuration space we try, the larger number of manifolds created. And in addition to the diagrams pointed out there appear diagrams describing pure polarisation effects Fig. 3e. and the creation of submonifolds (topology changes) Fig. 3f. I stress again that this difficulty appears not from the fact that the time variable is badly chosen but rather from that it is impossible to choose the time variable to fit all possible trajectories simultaneously, however the choice is made. This is a prerogative of a device which has to be described by a trajectory of its own in the same configuration space and has to enforce us to choose the time variable properly.

Thus, we come up to the need to consider a "multi-particle"(or complex topology) theory. In QFT this is achieved by second quantization of a one-particle wave function. In quantum gravity that is called third quantisation.

4. Third quantization

The procedure of second quantization which is used in QFT cannot straightforwardly be adopted to third quantisation. Indeed, in particle physics trajectories of a particle are just one-dimensional lines and the only way to reverse the trajectory backward with time is to reverse the particle as a whole. This would correspond to third quantization in the framework of minisuperspace models which describes creation and annihilation processes of whole universes. On the contrary, in quantum gravity we have to admit the possibility when just a piece of our spatial manifold is reversed that is a small folding from the spacetime point of view. Therefore, we have to reserve the possibility to create an arbitrary small submanifold and even a particular point in a limit.

We start first with the last case. For the sake of simplicity we shall use a lattice approximation of the coordinate manifold S . So the coordinates x will take discrete values with an interval Δx which afterward we have to tend to zero. Then the minimal size of the spatial manifold to be created is evidently $\Delta V = (\Delta x)^3$. Further, we shall call such a manifold as an elementary cell of our space. The configuration space of the cell is $n+1$-dimensional manifold M_n which has been introduced in the previous section . Quantum states of the cell may be described by a local wave function $\Psi_x(z)$ which has an additional label x pointing out the point of S at which the cell is placed. This function has to obey one of the local WDW equations (3.11). In the case under consideration in virtue of the absence of a potential term the local WDW equation is the ordinary finite-dimensional wave equation on M_n for a massless field.

Let us now assume that the number of such cells may be a variable. This means that at a particular supporting point of the coordinate manifold $x \in S$ there is a number of elementary cells corresponding to the physical space. In quantum theory this fact is accounted by third quantization of the local wave function Ψ_x introduced above. The last one becomes field operators and can be expanded in the form (for simplicity we consider Ψ_x to be a real scalar function)

$$\Psi_x = \sum_p C(p,x)f(p,x) + C^+(p,x)f^*(p,x), \tag{4.13}$$

where $f(p,z) = (2\omega_p(2\pi)^n)^{-\frac{1}{2}}e^{-i\omega_p z^0 + ipz}$ (here $\omega_p = |p|$ and we put $\Delta V = 1$) is the set of positive frequency solutions to the local WDW equation and the operators $C(p,x)$ and $C^+(p,x)$ satisfy the standard commutation relations

$$[C(p,x), C^+(p',y)] = \delta_{pp'}\,\delta(x,y). \tag{4.14}$$

The field operators Ψ_x act on a Hilbert space of states which has well known structure in Fock representation. The vacuum state is defined by the relations $C(x,p) \mid 0 >= 0$ (for all $x \in S$), $< 0|0 >= 1$.

Acting by the creation operators $C^+(p,x)$ on the vacuum state we can construct states describing a universe of an arbitrary spatial topology. In particular, the states describing the ordinary universe have the structure

$$|f >= \sum_{[p(x)]} F_{p(x)}|1_{p(x)} >, \qquad |1_{p(x)} >= \frac{1}{Z_1}\prod_{x\in S} C^+(x,p(x))|0 >, \tag{4.15}$$

where Z is a normalization constant and the wave function describing a simple universe takes the form

$$< 0|\Psi|f >=< 0|\prod_{x\in S}\Psi_x|f >= \sum_{[p(x)]} F_{p(x)}f_{p(x)} \tag{4.16}$$

where $f_{p(x)} = \prod_{x\in S} f(p(x), x)$ coincide with the positive frequency solutions (3.12). The states describing a universe with n disconnected spatial components have the following structure

$$|n>=|1_{p_1(x)},...,1_{p_n(x)}>=\frac{1}{Z_n}\prod_{i=1}^{n}\prod_{x\in S}C^+(x,p_i(x))|0> \tag{4.17}$$

(we remind that in the model under consideration due to the existence of l_{min} the coordinates x take discrete values). Besides these states describing simplest topologies the considered approach allows to construct nontrivial topologies as well. This is due to the fact that the tensor product in (4.15), (4.17) may be defined either over the whole coordinate manifold S or over a part of it $K \subset S$. In this manner, taking sufficiently small pieces K_i of the coordinate manifold S we can glue arbitrarily complex physical spaces. In order to construct the states of such a kind it turns out to be convenient to introduce the following set of operators

$$a(K,p(K))=\prod_{x\in K}C(x,p(x)),\qquad a^+(K,p(K))=\prod_{x\in K}C^+(x,p(x)). \tag{4.18}$$

These operators have the clear interpretation, e.g. the operator $a^+(K,p(K))$ creates the whole region $K \in S$ having the quantum numbers $p(K)$. Thus, in the general case states of the universe will be described by vectors of the type

$$|\Phi>=c_0|0>+\sum_I c_I a_I^+|0>+\sum_{I,J}c_{IJ}a_I^+a_J^+|0>+.... \tag{4.19}$$

5. Interpretation and observables

Now consider the interpretation of the suggested approach. Ordinary measurements are usually performed only on a part K of the coordinate manifold S. There are two possibilities. The first one is when an observer measures all of the quantum state of the region K and the second more probable one is when the observer measures only a part of the state. In the second case the observer considers K as if it were a part of the ordinary flat space. Therefore, the part of the quantum state which will be measured, appears to be in a mixed state. This means the loss of quantum coherence widely discussed in Refs.[17]. In order to describe measurements of the second type we define the following density matrix for the region K

$$\rho^{nm}(K)=\frac{1}{N(K)}<\Phi|a^+(K,n(K))a(K,m(K))|\Phi>, \tag{5.20}$$

where $|\Phi>$ is an arbitrary state vector of the (4.19) type and $N(K)$ is a normalization function which measures the difference of the real spatial topology from that of the coordinate manifold S. If we consider the smallest region K which contains only one point x of the space S the normalisation function $N(x)$ in (5.20) will play the role of a "density" of the physical space. For the states (4.15), (4.17) we have $N(x) = 1$ and $N(x) = n$ respectively. Thus, if $A(K)$ is any observable we find $< A >= \frac{1}{N} Tr(A\rho)$.

To describe complete measurements (of the first type) let us consider creation operators for localised states of the metric field. These operators is constructed in analogy with the well known Newton-Wigner operators [42] (see, also, [15])

$$\varphi^+(x,z) = \sum_p \psi_p^*(z)\, b_{x^0}^+(x,p) \ ,$$

$$\varphi^+(K, O(K)) = \sum_{p(K)} \psi_{p(K)}^*[z(K)]\, b_{x^0}^+(K, p(K)) \ ,$$

where $\psi_p = (2\pi)^{-\frac{n}{2}} e^{ipz - i\omega_p z^0}$ and $\psi_{p(K)}^*[O(K)] = \prod_{x\in K} \psi_{p(x)}^*(O(x))$ and the operators $b_{x^0}^+(K, p(K))$ is defined in the same manner as in Eq. (4.18). Thus, the state $|x, z\rangle = \varphi^+(x, z)|0_x\rangle$ describes a unique point of the physical space with configuration variables localised at the point $O = (z^0, z^n)$ of a superspacelike hypersurface $\Sigma_x^0 \subset M_x$ while $|K, z(K)\rangle = \varphi^+(K, O(K))|0_x\rangle$ describes a whole region $K \subset S$ with field variables $O(K)$.

Thus, the probability to find a unique point x with gravitational and scalar field variables being at the point O of the superspacelike hypersurface Σ_x^0 may be determined as

$$dW_x = P_x(z) d\Sigma_x^0 \ , \tag{5.21}$$

where $P_x(z)$ is the one point probability density

$$P_x(O) = |\langle x, z|\Phi\rangle|^2 \ , \tag{5.22}$$

with $|\Phi\rangle$ being an arbitrary initial state (4.19). In the case of the whole region K the expression (5.21) is generalised in a usual manner

$$dW_K = P_K(z(K)) D\Sigma_K^0 \ , \tag{5.23}$$

where $D\Sigma_K^0 = \prod_{x\in K} d\Sigma_x^0$ is the volume element on the hypersurface $\Sigma_K^0 \subset M_K \ : \ z^0(x) = z^0$, and $P_K = |< K, z(K)\ |\Phi >|^2$.

6. Topology fluctuations and quantum creation of a quasi-homogeneous inflationary Universe

Let us consider an automodel solution describing an infalationary Universe. The inflationary stage in the evolution of the Universe begins under the following conditions

$$W \approx 2\Lambda = const, \tag{6.24}$$

which imply the potential becomes an effective cosmological constant [45, 43]. The conditions (6.24) imply also defined restrictions on the degree of inhomogeneity of the Universe. In this case the Wheeler-DeWitt equation can be read off ($g = e^{3x^0}$)

$$(\Delta_x + 12\Lambda e^{3x^0(x)})\Psi = 0, \qquad x \in S. \tag{6.25}$$

where Δ_x denotes the same Laplace operator as in (3.11). The distinctive feature of this equation is the fact that it has an explicit "time"-dependent form. Therefore, one could expect the existence of quantum polarisation effects (topology fluctuation or the so-called spacetime foam [49]). These effects can be calculated either by singling out the asymptotic *in* and *out* regions on the configuration space M for which we can determine positive-frequency solutions to Eq.(6.25) (see for example [23]), or by using the diagonalisation of the Hamiltonian technique [15] by means of calculating depending on time Bogoliubov's coefficients.

Let us consider solutions to an arbitrary local x-equation (6.25). These solutions can be represented in the form $u(p,x) = (2\pi)^{-\frac{3}{2}} e^{ipx}\varphi_p(x^0)$ where φ satisfies the equation

$$\frac{d^2\varphi_p}{d^2x^0} + \omega_p^2(x^0)\varphi_p = 0, \qquad \omega^2 = p^2 + 6\Lambda e^{3x^0} \tag{6.26}$$

and is expressed in terms of Bessel or Hankel functions. The function φ can be decomposed in positive and negative frequency parts

$$\begin{aligned} \varphi_p &= \frac{1}{\sqrt{2\omega_p}}(\alpha_p e^{i\theta_p} + \beta_p e^{-i\theta_p}) \ , \\ \frac{d\varphi_p}{dx^0} &= i\sqrt{\frac{\omega_p}{2}}(\alpha_p e^{i\theta_p} - \beta_p e^{-i\theta_p}) \ , \end{aligned} \tag{6.27}$$

where $\theta_p = \int^{x^0} \omega_p dx^0$. The functions α_p and β_p satisfy identity $|\alpha_p|^2 - |\beta_p|^2 = 1$ and define the depending on time Bogoliubov coefficients [15]. The depending on time creation and annihilation operators take the form

$$\begin{aligned} b_{x^0}(x,p) &= \alpha_p(x^0)C(x,p) + \beta_p^*(x^0)C^+(x,p) \ , \\ b_{x^0}^+(x,p) &= \alpha_p^*(x^0)C^+(x,p) + \beta_p(x^0)C(x,p) \ . \end{aligned} \tag{6.28}$$

In terms of these operators the super-Hamiltonian of the field Ψ_x (the Hamiltonian density) becomes diagonal

$$E_x = \int_{\Sigma^0} \Theta_{0A} d\Sigma_x^A = \frac{1}{2}\sum_p \omega_p\left(x^0\right)\left(b_{x^0}^+(x,p)\, b_{x^0}(x,p) + b_{x^0}(x,p)\, b_{x^0}^+(x,p)\right) \ , \tag{6.29}$$

where $\Theta_{AB} = \nabla_A\Psi_x\nabla_B\Psi_x - \frac{1}{2}G_{AB}\left(\nabla_C\Psi_x\nabla^C\Psi_x - (U+\xi P)\Psi_x^2\right)$ and $d\Sigma_x^0 = d^nx$. The ground state of the Hamiltonian is determined by the conditions $b_{p}(x,p)|0_{x}\rangle = 0$ for all x and p and is also depending on time. The excitations of (6.29) are interpreted as points of physical space having the coordinate $x \in S$.

Now we determine two asymptotic regions as *in* $(z^0 \to -\infty)$ and *out* $(z^0 \to +\infty)$. In these regions the functions α_p and β_p take constant values and therefore, in these regions we can define positive frequency modes as $U(p,x) = (2\omega_p(2\pi)^n)^{-\frac{1}{2}}e^{ipx - i\theta_p}$. Substituting the initial conditions $\alpha_p = 1$, $\beta_p = 0$ as $z^0 \to -\infty$ in (6.26), (6.27) we find that in the *out* region the Bogoliubov coefficients are

$$\alpha_p = (\exp(\frac{3\pi p}{2})/2sh(\frac{3\pi p}{2}))^{\frac{1}{2}}, \qquad \beta_p = (\exp(-\frac{3\pi p}{2})/2sh(\frac{3\pi p}{2}))^{\frac{1}{2}}. \tag{6.30}$$

Then, for example, if the initial state of the "superspace"-Hamiltonian is the ground state $|0_{in}>$ in the *out* region the density matrix (5.20) takes form

$$\rho^{pq}(K) = \prod_{x\in K}\rho^{p(x)q(x)}(x), \tag{6.31}$$

where $\rho(x)$ is a one-point density matrix

$$\rho^{pq}(x) = \frac{1}{N(x)}|\beta_p|^2\delta(p,q) = \frac{1}{N(x)}\frac{1}{e^{3\pi p}-1}\delta(p,q). \tag{6.32}$$

The normalization function in (6.32) is given by $N(x) = V_x c_n$, where V_x is the spatial volume of the configuration space M_x and c_n is a constant $c_n = \frac{2\Gamma(n)\xi(n)}{6^n\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}$ here $\xi(s) = \sum_{k=1}^{\infty}k^{-s}$ is the Riemann ξ function. In the case $n=2$ or $n=3$ we find $c_2 = \frac{\pi}{108}$ and $c_3 = \frac{2\xi(3)}{81\pi^2}$ respectively. The matrix (6.31) does not depend on spatial coordinates and has the Plankian form with the temperature $T = \frac{1}{3\pi}$ and therefore, we obtain the creation of a universe which in average turns out to be homogeneous.

To conclude this section we note that the property of the created universe to be homogeneous follows in the first place from the specific choice of the homogeneous initial quantum state $|0_{in}>$. Nevertheless, the considered model shows that during the evolution topology fluctuations strongly increase. Indeed, in the *out* region the "space density" $N(x)$ turns out to be proportional to the spatial volume V_x of the configuration manifold M_x. In the given model the volume V_x is infinite and that of the "space density" but it would not be so if we consider the real potential in (6.25) (or consider next orders of an approximation procedure) and, therefore, one could expect the value $N(x)$ to be sufficiently large but finite. Then if the initial state corresponds to a simple universe (4.15) having $N_{in}(x) = 1$ the final state will be described by the density matrix (6.31) up to the order of $1/V$.

7. On a Modification of the Ordinary Field Theory

As was shown above the numbers $N(x)$ vary during the evolution. This means that the foamy structure of the physical space is not fixed and is determined dynamically. In this section we shall discuss an interesting possibility when the spatial continuum has "hollows" at small distances (i.e. $N(\mathbf{k}) \to 0$ if $\mathbf{k} \to \infty$, where $N(\mathbf{k}) = (2\pi)^{-2/3} \int N(x) \exp(-i\mathbf{k}x) d^3x$) which may be used to overcome the divergences problem in conventional quantum gravity. As an example, we consider now a free massless scalar field φ.

In terms of Fourier expansion for φ

$$\varphi(x,t) = (2\pi)^{-2/3} \int \frac{d^3\mathbf{k}}{\sqrt{2k}} \left\{ A(\mathbf{k}) e^{i\mathbf{k}x - ikt} + A^+(\mathbf{k}) e^{-i\mathbf{k}x + ikt} \right\} \tag{7.33}$$

(here $k = |\mathbf{k}|$), the field Hamiltonian takes the form of a sum of independent non-interacting oscillators

$$H = \frac{1}{2} \int k \left[A(\mathbf{k}) A^+(\mathbf{k}) + A^+(\mathbf{k}) A(\mathbf{k}) \right] d^3\mathbf{k}. \tag{7.34}$$

Since the density of the physical space $N(\mathbf{k})$ is a variable quantity so does the number of field oscillators. This fact may be accounted for in a phenomenological manner by introducing creation and annihilation operators of the field oscillators which obey the same (anti) commutation relations as in (4.14):

$$[C(\mathbf{k},n), C^+(\mathbf{k}',m)]_\pm = \delta_{n,m} \delta^3(\mathbf{k} - \mathbf{k}') \tag{7.35}$$

where dependence of the operators on the quantities $\mathbf{k}$ and n is connected with the classification of the states an individual oscillator (the spectrum of the oscillator has the form $\epsilon(\mathbf{k},n) = kn + \epsilon_0(\mathbf{k})$, where the quantity $\epsilon_0(\mathbf{k})$ gives the contribution of vacuum fluctuations of the field). In the vacuum state $|0\rangle$ (which is determined now by $C(\mathbf{k},n) \mid 0 >= 0$) field oscillators (and all field observables) are absent. The operator of total energy of the field can be generalized in a natural way as

$$E = \sum \epsilon(\mathbf{k},n) C^+(\mathbf{k},n) C(\mathbf{k},n). \tag{7.36}$$

The connection with the standard field variables can be determined with the help of operators which increase (decrease) the energy of system on k ($[E, A^{(+)}(\mathbf{k})]_- = \pm k A^{(+)}(\mathbf{k})$)

$$A^+(\mathbf{k}) = \sum_{n=0}^{\infty} (n+1)^{1/2} C^+(\mathbf{k}, n+1) C(\mathbf{k}, n), \tag{7.37}$$

$$A(\mathbf{k}) = \sum_{n=0}^{\infty} (n+1)^{1/2} C^+(\mathbf{k}, n) C(\mathbf{k}, n+1). \tag{7.38}$$

It can be seen from (7.36)-(7.38) that the operators A and A^+ satisfy the commutation relations

$$[A(\mathbf{k}), A^+(\mathbf{k}')]_- = N(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'), \tag{7.39}$$

where $N(\mathbf{k}) = \sum_{n=0}^{\infty} C^+(\mathbf{k},n)C(\mathbf{k},n)$ is the complete number of spatial domains related to the wave number $\mathbf{k}$. If one restricts oneself by the states [of the type (4.15)] with $N(\mathbf{k}) = 1$, the operators $A^+(\mathbf{k})$ and $A(\mathbf{k})$ certainly coincide with the standard creation and annihilation operators of scalar particles.

As was shown the quantities $N(\mathbf{k},n) = C^+(\mathbf{k},n)C(\mathbf{k},n)$ must be determined by dynamics. However, they can be estimated from simple considerations. It is clear that in the absence of the gravitational interaction the quantities $N(\mathbf{k},n)$ remain constant. Then, for instance, under the assumption of bounded density $N < \infty$ of oscillators satisfying the Fermi statistics it is easy to find that the occupation numbers corresponding to the ground state are

$$N(\mathbf{k},n) = \theta(\mu - \epsilon(\mathbf{k},n)), \tag{7.40}$$

where $\theta(x) =[\ 0$ for $x < 0$ and 1 for $x > 0]$, and μ is determined via the total number of oscillators $N = \sum N(\mathbf{k},n)$. Using (7.40), one can found the number of oscillators corresponding to a wave vector $\mathbf{k}$ as

$$N(\mathbf{k}) = \sum_{n=0}^{\infty} \theta(\mu - \epsilon(\mathbf{k},n)) = [1 + (\mu - \epsilon_0(\mathbf{k}))/k], \tag{7.41}$$

here $[x]$ denotes the entire part of x. In particular, one can see from (7.41) that $N(\mathbf{k}) = 0$ for $\mu < \epsilon_0(\mathbf{k})$.

For the excited states formed by the action of the operators $A^+(\mathbf{k})$ on the ground state (7.40), the operator $N(\mathbf{k})$ is the usual function (7.41). Let us consider excitations of the field (scalar particles) described by the thermal equilibrium state corresponding to temperature T (one could expect that the spatial domains created near the singularity have a thermal spectrum [23]). Then the correlation function for the potentials of the field (7.33) takes form

$$< \varphi(x)\varphi(x+r) >= (2\pi^2)^{-1} \int \Phi^2(k) \frac{\sin kr}{kr} \frac{dk}{k}, \tag{7.42}$$

where $\Phi^2(k) = k^2 N(k) \frac{1}{2} \coth(\frac{k}{2T})$. In the wave number range $k \ll (T,\mu)$ the spectrum of the field fluctuations is scale-independent: $\Phi^2(k) \approx TkN(k) = T\mu$.

We also note that the ground state determined by the occupation numbers (7.40) has a bounded energy density of the field which can be considered as a "dark matter". In addition, we note that the above property of spectrum to be scale-invariant at large scales for the thermal equilibrium state, actually, does not depend on the statistics of the oscillators (i.e., upon the sign $\pm$ in (7.35), (4.14)).

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