Chapter 48

Introduction. The Current State of the Universe

Current classical cosmology succesfully describes the main features of the universe, but uses ford that some specific initial conditions. In the framework of classical cosmology these conditions do not have thier own reasonable explanation. They are just selected in such way that the theoretical predictions be compatible with the actual observations. A more deeper understanding of why the universe has these and not the other properties can be provided by quantum cosmology. The most important unsolved issue is the nature of the cosmological singularity whose existence follows from classical general relativity. The phenomenon of singularity is probably the most compelling reason for replacing classical cosmology with quantum one.

Let us recall some properties of the actual world which seem to have their origin in the very early universe (see standard text books [1] [2] [3] [4]).

The distribution of galaxies in space as well as the distribution of their red shifts indicate that at largest scales the universe is homogeneous and isottropic. The most convincing manifestation of the large scale homogeneity and isotropy of the universe is the absence of the angular variations of the temperature of the microwave background radiation: $\Delta T/T <_{\sim} 10^{-4}$. All the observational data pont to the conclusion that in the first approximation the overall structure and dynamics of the universe can be described by the Friedmann (or Friedmann-Robertson-Walker, FRW) line element,

$$ds^2 = c^2\, dt^2 - a^2(t) dl^2. \tag{1}$$

It is known that the spatial part if the metric 1 dl^2, can correspond to the closed ($k = +1$), open ($k = -1$), or flat ($k = 0$) 3-dimensional spaces. The actual sign of the space curvature depends on the ratio $\Omega = \rho_m/\rho_c$ of the mean matter density ρ_m to the critical density $\rho = \frac{3}{8\pi G} H^2$ where H is the Hubble parameter H(t), $H(t) \equiv \dot{a}/a$. The majority of the avaiable astronomical data favour the value $\Omega < 1$ which implies that $k = -1$. However the observations can not presently exclude neither $k = 0$ nor $k = +1$. In any case the current value of the parameter Ω seems to be very close to the unity.

Although the overall structure of the universe is homogeneous and isotropic, it is obviously not the case at scales characteristic of galaxies and their clusters. It is believed that the smaller scale inhomogeneties were formed as a result of growth of small initial perturbations. In order to produce the observed inhomogenieties the initial perturbations must have the specific amplitude and specific spectrum. There are some theoretical and observational arguments in support of the so-called "flat "Harrison-Zeldovich spectrum [5] [6] of the initial fluctuations.

The dynamical characteristics of the averaged distribution of matter, the growth and formation of the smaller scale inhomogeneities, the abundances of various chemical elements, as well as other features of the actual universe are successfully brought together by the "standard "classical cosmological theory. The trouble is, however, that the "standard "theory postulates certain properties of the universe rather than derives them from more fundamental principles. For instance, the observational fact of the angular uniformity of the temperature of the microwave background radiation over the sky does not have another rational explanation except being a consequence of the postulated, everlasting homogeneity and isotropy. A more natural explanation to a set of observational fact is provided by the inflationary hypothesis [7] [8] [9].

According to the inflationary hypothesis the spatial volume of the universe confined to the current Hubble distance $l_H = c/H \approx 10^{28} cm$ or, possibly, even much larger volume has developed from a small region which was causally connected in the very distant past. If the inflationary stage in the evolution of the very early universe did really take place then the large scale homogeneity and isotropy as well as the closeness of Ω to the unity can be explained as the consequences of the inflationary expansion.

The most popular model for the inflationary stage of expansion is provided by the De-Sitter solution. Originally it was derived as a solution to the vacuum Einstein equations with a constant cosmological Λ-term. However, it can also be treated as a solution to the Einstein equations with matter source satisfying the effective equation of state $p = -\epsilon$. The De-Sitter solution describes a space-time with constant 4-curvature. This space-time is a symmetric as Minkowski space-time, i.e. it admits a 10-parameter group of motion. The line element of the De-Sitter space-time has the form (see, for example, [10]):

$$ds^2 = c^2\, dt^2 - a^2(t) \left[dr^2 + sin^2 r \left(d\theta^2 + sin^2\theta\, d\psi^2\right)\right], \quad (2)$$

where $a(t) = r_0 cosh(ct/r_0)$, and $r_0 = const$. It is known that in the De-Sitter space-time one can also introduce the frames of reference with flat or open (hyperbolic) space sections. These coordinate systems do not cover the whole of De-Sitter space-time. The often used is the flat 3-space representation:

$$ds^2 = c^2\, dt^2 - a^2(t) \left(dx^2 + dy^2 + dz^2\right), \quad (3)$$

where $a(t) = exp(H_0 t)$, and $H_0 = const$. The constant H_0 plays a role of the Hubble constant at the De-Sitter stage of expansion. The scale factor $a(t)$ of the line element 3 approaches the behavior $a(t) \sim exp(H_0 t)$ very quickly during several characteristic time intervals $t = r_0/c$.

In order to see the advantages of the inflationary expansion let us assume that at $t = 0$ the distance between two idealized physical objects was of order of a few Planckian scales, $l_{Pl} = 10^{-33} cm$. Then, it can be shown that the present day the distance between these objects can be as large as the present day Hubble distance $l_H \approx 10^{28} cm$, if the duration of inflationary stage Δt was sufficiently long, $H_0 \Delta t >\sim 65$. In this way a small causally connected region could have been grown to the size of the presently observed universe.

Chapter 49

A Complete Cosmological Theory

The inflationary stage help us to resolve several comological puzzles. However, the origin of the inflationary stage needs to be explained. The question still remains what kind of evolution dis the universe experience before the inflationary stage and how did the universe itself originate. A frequently made assumption is that prior to the De-Sitter stage there was a preceeding radiation-dominated era. This assumption just postpones the answer to the above mentioned questions and inevitably returns us to the problem of cosmological singularity and quantum gravity. As a cardinal solution to the problem, it was suggested [11] [12] that the inflationary era was preceeded by an essentially quantum-gravitational phenomenon called a spontaneous birth of the universe. A theory capable of describing the classical stages of evolution of the universe, as well as its quantum-gravitational origin was called a complete cosmological theory. Let us present the main features of such a theory.

The desired evolution of the scale factor $a(t)$ is shown in Fig. 1. According to this scenario the moment of appearance of the classical universe corresponded to $t = 0$. After the moment of time the inflationary evolution started and was described by eq.2. It is natural to expect that all the characteristic parameters of the newly born universe were of order of the Planckian scales, i.e. the classical space-time came into being near the limit of applicability of classical general relativity. The inflationary expansion is able to pickup such a micro-universe and to increase its size up to the necessary value. The wave-lime joining the points $a = 0$ and $a = l_{Pl}$ at Fig. 1 was meant to describe the essentially quantum-gravitational process reminiscent of quantum tunneling or quantum decay, and resulted in the nucleation of the universe in the state of the classical De-Sitter expansion. It is reasonable to suppose that at the beginning of classical evolution the deviations from highly symmetric De-Sitter solution were negligibly small. Moreover, it seems to be sufficient to take these deviations with the minimally possible amplitude, i.e. at the level of quantum zero-point fluctuations. During the inflationary period these fluctuations could have been amplified and produce the density perturbations and gravitational waves. The density perturbations are needed to form the observed inhomogeneities in the universe. Gravitational waves seem to be the only source of impartial information about the inflationary epoch and the quantum brth of the universe. These matters will be discussed in more detail below. We will see how the notions introduced above will acquire more precise formulation.

Chapter 50

An Overview of Quantum Effects in Cosmology

From this brief exposition of a complete cosmological theory it is clear that quantum effects and quantum concepts should play a decisive role in different context and at different levels of approximation. It is useful ti give a short classification of the areas of further discussion where the quantum notions will be dealt with. It is worth emphasizing that below we will often use the common and powerful technique wich is the splitting up of a given problem into the "background "and "perturbational "parts.

We will start from the description of classical perturbations atr a classical background space-time. The physical meaning of such an effect as parametric amplification of gravity-wave perturbations can be clearly seen already at this level of approxiamation. The next level of approximation treats the perturbations as the quantized fields interacting with the classical background geometry. At this level of approximation we will discuss the graviton creation in the homogeneous and isotropic universe. The main attention will be paid to the actual quantum state of created gravitons. It will be shown that it is so-called squeezed quantum state.

At a still deeper level, the background geometry and matter fields are also treated quantum-mechanically — this is the realm of quantum cosmology. The main object of interest in quantum cosmology is the wave function of the universe which, in general, describes all degrees of freedom at the equal footing. This level of discussion is appropriate for tackling such issues as the beginning and the end of classical evolution as well as quantum birth of the universe. However, there is no one unique wave function of the universe, there are many of them. All possible wave functions constitute the whole space of the wave functions. One can introduce the notion of a Wave Function given in the space of all possible wave functions. In order words, a wave function of the universe becomes an operator acting on the Wave Function describing the many universes system. This is the subject of the now very popular so-called third-quantized theory. It is aimed at describing the multiple production and annihilation of the baby-universes. This fascinating subject is still very unclear and is beyond the scope of the present lectures. The reader is reffered to the recent review and technical papers on the subject [13] [14] [15] [16] [17] [18]. In some sense the different theories listed from above to the bottom are various approximation to the theories listed in the opposite direction.

Let us start from the classical theory of small perturbations superimposed on a given background space-time.

Chapter 51

Parametric (Superadiabatic) Amplification of Classical Gravitational Waves

We will consider classical weak gravitational waves. The main purpose is to study the parametric (superadiabatic) amplification of gravitational waves [19]. We assume that the space-time metric $g_{\mu\nu}$ can be presented in the form

$$g_{\mu\nu} \approx g_{\mu\nu}^{(0)} + h_{\mu\nu},$$

where $g_{\mu\nu}^{(0)}$ is the background metric:

$$ds^2 = a^2(\eta)\left(d\eta^2 - dx^2 - dy^2 - dz^2\right) \tag{1}$$

and $h_{\mu\nu}$-gravitational wave perturbations. The small correections $h_{\mu\nu}$ can be simplified by using the avaiable gauge freedom. The remaining components can be decomposed into the mode functions, so that for a given mode one has

$$h_i^k = \frac{1}{a}\mu(\eta)G_i^k(x,y,z). \tag{2}$$

In the case under discussion the eigenfunctions G_i^k can be taken in the simplest form:

$$G_i^k \sim exp[\pm i(n_1x + n_2y + n_3z)], n_1^2 + n_2^2 + n_3^2 = n^2.$$

The main equation to be solved is

$$\mu'' + \mu(n^2 - a''/a) = 0, \tag{3}$$

where $' = \frac{d}{d\eta}$ and n is the wave-number, the wave lenght being $\lambda g = \frac{2\pi a}{n}$.

Equation 3 has the form of an equation for the oscillator with the varrying frequency, that is we deal with a parametrically exited oscillator. The same eq. 3 can be regarded as the Schrödinger equation for a particle with energy n^2 and the potential $U(\eta) = a''/a$.

In the regime $n^2/gg|U(\eta)|$ the solutions to eq. 3 have the form $\mu \sim e^{\pm in\eta}$ so that one has usual high-frequency waves with adiabatically diminishing amplitude: $h \sim \frac{1}{a} sinn\eta + \phi$. However, in the regime $n^2 \ll |U(\eta)|$ there are two other solutions to the second-order differential eq. 3 wich are $\mu_1 \sim a$ and $\mu_2 \sim a \int a^{-2} d\eta$. A typical potential $U(\eta)$ is shown in Fig. 2. The waves with $n^2 \gg |U(\eta)|$ for all values of η have adiabatically decreasing amplitude and are shown symbolically by the wavy line above the potential in Fig. 2.

The waves satisfing $n^2 \ll |U(\eta)|$ for some η encounter the potential barrier and are governed by the solutions μ_1 and μ_2 in the under-barrier region. It can be shown that the dominant solution is μ_1. Indeed, the amplitude of the wave after its exit from under the potential barier depends, in general, on the initial phase ϕ and on both solutions μ_1 and μ_2. However, the averaging over the initial phase ϕ pf the wave leads to a dominant contribution from μ_1. This means that the adiabatic factor $1/a$ is cancelled out by $\mu_1 \sim a$ and, as a result, the physical amplitude of the wave h, eq. 2, does not change in th region occupied by the barier, instead of diminishing adiabatically . Thus, the actual final amplitude of the wave h_f is larger than it would have been, if the wave always behaved adiabatically, and it is equal to the initial amplitude h_i, $h_f \approx h_i$. This is the essence of the mechanism of the superadiabatic amplification of gravitational waves.

The amplification coefficient is just the ratio $a(\eta_f)/a(\eta_i)$ where $a(\eta_f)$ is the value of the scale factor at the moment of exit of the wave from under the potential, and $a(\eta_i)$ is the value of the scale factor at the moment of entering the under-barrier region. It is seen from Fig. 2 that different waves, that is the waves with different wave numbers n, stay under the potential for different duration of time. In other words, the amplification coefficient is determined by the form of the potential and depends on n. This leads to the transformation of the initial spectrum of the waves into the final one.

Until now we were discussing the amplitude of classical waves. The initial amplitudes and the spectrum of the waves could be arbitrary. Now, remaining at the same classical level, we will discuss the amplification of the zero-point quantum fluctuations. This means that the strictly quantum-mechanical notion of the vacuum state for each mode we will replace with the classical waves with certain amplitudes and spectrum. intuitively, the vacuum state corresponds to the idea of having 1/2 of a quantum in each mode. The energy density of gravitational waves scales as $\epsilon_g \sim \frac{c^4 h^2}{G\lambda^2}$. For a given wavelenght λ we want to have energy $\frac{1}{2}\hbar\omega$ in the volume $\sim \lambda^3$. In this way we derive the vacuum amplitude of gravitational waves with the wavelenght λ, wich turns out to be equal to $h(\lambda) = \frac{l_{Pl}}{\lambda}$. Hence, the initial vacuum spectrum of gravitational waves scales as $h(n) \sim n$. This is the spectrum to be transformed by the interaction with an external gravitational field. The amplification process will make the number of quanta in the mode much larger than 1/2.

Chapter 52

Graviton Creation in an Inflationary Universe

One can immediately apply the given considerations to the inflationary scenario. In terms of η-time the De-Sitter solution (3) has the scale factor $a(\eta) \sim -1/\eta$. (It is convenient to have η negative and growing from $-\infty$.) We assume that the De-Sitter stage ends at some $\eta = \eta_1 < 0$ and goes over into the radiation-dominated stage with $a(\eta) \sim \eta$. Then the relevant potential $U(\eta)$ has the form shown in Fig. 3 by a solid line 1. A given wave with the wave number n enters the potential at some η_i, when $n^2 \approx a''/a$, which leads to the entering condition $\eta\eta_i \approx 1$. For different waves this condition is satisfied at different $a(\eta_i)$. All the waves leave the potential roughly at the same η_f which corresponds to the same scale factor $a(\eta_f)$. The amplification coefficient scales as

$$\frac{a(\eta_f)}{a(\eta_i)} = \frac{\eta_i}{\eta_f} = \frac{n\eta_i}{n\eta_f} \sim \frac{1}{n}.$$

Multiplying the initial vacuum spectrum $h_i(n) \sim n$ by the amplification factor $\sim \frac{1}{n}$ one obtains the final amplitude which is independent of n: $h_f(n) \sim n^0$.

To make this part of the discussion more precise let us consider a concrete example. Let the scale factor at the inflationary stage be

$$a(\eta) = -\frac{1}{H_0\eta},$$

where H_0 is the Hubble constant at the De-Stter stage. At some $\eta = \eta_1 < 0$, $a(\eta)$ joins to the scale factor of the radiation dominated stage:

$$a(\eta) = \frac{1}{H_0\eta_1^2}(\eta - 2\eta_1).$$

At the joining point $\eta = \eta_1$ the values of $a(\eta)$ as well as their first η-derivatives coincide. In the region $-\infty < \eta \leq \eta_1$ the general solution to eg. (6) has the form

$$\mu_i = A\left[\cos(n\eta + \phi) + \frac{1}{n\eta}\sin(n\eta + \phi)\right]$$

where A and ϕ are arbitrary constants. For $+\infty > \eta \geq \eta_1$ the general solution has the form

$$\mu_e = B\sin(n\eta + \chi), \tag{1}$$

where B and χ are to be determined from the joining conditions for μ and μ' at $\eta = \eta_1$.

It is convenient to introduce the notation $n\eta_1 \equiv x$. Then it can be shown from the joining conditions that

$$\begin{aligned}\left(\frac{B}{A}\right)^2 &= \left(1 + \frac{1}{x^2}\right)\cos^2(x + \phi)\\ &+ \left(1 + \frac{3}{x^2} + \frac{1}{x^4}\right) - \frac{2}{x^3}\sin(x + \phi)\cos(x + \phi).\end{aligned}$$

It is seen that $\left(\frac{B}{A}\right)^2$ depends on the initial phase ϕ. however, the averaging over the phase ϕ always leads to the superadiabatic amplification:

$$\left(\overline{\left(\frac{B}{A}\right)^2}\right)^{1/2} = \left(1 + \frac{2}{x^2} + \frac{1}{x^4}\right)^{1/2}.$$

Since $x \ll 1$ this quantity is much larger than the unity:

$$\left[\overline{\left(\frac{B}{A}\right)^2}\right] \approx \frac{1}{\sqrt{2}}\frac{1}{x^2} \gg 1 \tag{2}$$

Thus, at the left-hand side, i.e. for $\eta \to -\infty$, the solution is

$$h(n) \approx a^{-1} A(n) \cos(n\eta + \phi),$$

where $A(n) \sim n$. The amplitude of the last oscillation of the wave before entering the under barrier region is $h_i \approx \frac{H_0}{n} A(n)$. At the right-hand side, i.e. for $\eta \to +\infty$, the solution is

$$h(n) \approx a^{-1} B(n) \sin(n\eta + \chi).$$

By substituting here the "typical " value of B, derived from eq. (2), one obtains

$$h(n) \approx \frac{H_0\eta_1^2}{\eta - 2\eta_1} \frac{A(n)}{n^2\eta_1^2} \sin(n\eta + \chi) \tag{3}$$

The amplitude of the first oscillation of the wave, immediately after leaving the under barrier region, that is for $n(\eta - 2\eta_1) \approx 1$, is $h_f \approx \frac{H_0}{n} A(n)$. Hence, $h_i = h_f$ despite the huge increase of the scale factor.

We derived the typical soluiton (3) by using the condition that the initial phase ϕ was randomly disturbed. However, from this simple example one can also see that the cosmological expansion is the phase sensitive amplifier. This property is important for understanding of how the initial vacuum quantum state transforms into the final squeezed quantum state. We will consider this statement more rigorously later on. However, alredy now one can show that the two quadrature components of the wave, that is two coefficients in front of $\sin n\eta$ and $\cos n\eta$, respectively, behave differently. To see this, one can first find out the constant χ from the joining conditions and then rewrite the exact solution (1) in the form:

$$\begin{aligned}\mu_e &= A\left[-\frac{1}{x^2}\cos x\,\sin(x+\phi) + \frac{1}{x}\cos\phi - \sin\phi\right] \\ &+ A\cos n\eta\left[\frac{1}{x^2}\sin x\,\sin(x+\phi) + \frac{1}{x}\sin\phi + \cos\phi\right] \\ &\equiv A\,k_1\,\sin n\eta + A\,k_2\,\cos n\eta\end{aligned}$$

For $x \ll 1$ one has $k_1 \approx -\frac{1}{x^2}$ and $k_2 \approx 2\,\frac{1}{x}\,\sin\phi$. After squaring and averaging over the phase we obtain

$$\left(\overline{k_1^2}\right)^2 \approx \frac{1}{\sqrt{2}}\frac{1}{x^2}$$

$$\left(\overline{k_2^2}\right)^2 \approx \sqrt{2}\,\frac{1}{x}.$$

One can say that the variance of k_1 is much larger than the variance of k_2. Even more impressive is the difference between variances in the quadrature components $\sin n\overline{\eta}$ and $\cos n\overline{\eta}$, where $\overline{\eta} \equiv \eta - 2\eta_1$. In this case one has, correspondingly,

$$\left(\overline{k_1^2}\right)^{1/2} \approx 2.$$

Remember that before amplification, that is, for $\eta \to -\infty$, the variances were equal and small:

$$\left(\overline{k_1^2}\right)^{1/2} = \left(\overline{k_2^2}\right)^{1/2} = \frac{1}{\sqrt{2}}.$$

Let us return to the further evolution of the waves amplified during the inflationary period. the waves with different directions of propagation, i.e. with different wave vectors (n_x, n_y, n_z), are statistically independent. By averaging over statistical realizations one can derive the mean square amplitude of the amplified fluctuations by the end of the inflationary era:

$$< h^2 >= \int h_f^2(n)\,\frac{dn}{n}.$$

Since $h_f(n)$ does not depend on n, as was shown above, we arrive at the famous Harrison-Zeldovich "flat ". The equivalent definition of this spectrum is that every wave starts to decrease its amplitude from one and the same numerical value, independtly of n. Let us recall that the very notion of the amplitude of the wave assumes that the wave has completed at least one cycle of oscillations. In other words, it requires $n(\eta - 2\eta_1) \approx 1$ in equation (3). In the course of the further evolution the amplitude of the wave diminishes adiabatically.

In the time interval between the end of the inflationary era and the present epoch the shorter waves experienced a larger number of oscillations than the comparatively long waves. It means that the present day amplitude of the shorter waves is smaller than the amplitude of the longer waves. Another way of saying is that the shorter waves crossed the temporary Hubble radius and started to cool adiabatically earlier than the longer waves. While starting from the same amplitude, the shorter waves decreased more.

Since at the radiation dominated stage the scale factor goes as $a(\eta) \sim \eta$ the present-day amplitude of the waves scales as $h(n) \sim \frac{1}{n}$. It follows more formally from equation (3) for $\eta \to +\infty$.

In terms of the present-ady fequency ν, measured in Hz, the dependence is $h(\nu) \sim \nu^{-1}$. (This part of the spectrum was first discussed in [21].)

In the actual Universe the radiation-dominated epoch has changed to the matter-dominted epoch some time ago. At the matter-dominated stage the effective potential $U(\eta)$ in eq. (3) is again non-vanishing. This potential is shown by a solid line 2 in Figure 3. Short waves, i.e. the waves with sufficiently large n, do not interact with this potential. However, the longer waves (their contemporary frequencies lie in the interval $10^{-16}\ Hz <_{\sim} \nu <_{\sim} 10^{-18}\ Hz$ [20]) did encounter this potential and additionally transformed their spectrum. Since, at the matter-dominated stage, the scale factor goes as $a(\eta) \sim \eta^2$, the waves interacted with this potential have the present-day spectrum in the form $h(\nu) \sim \nu^{-2}$. (This part of the spectrum was first discussed in Ref. [22], see also [23], [24], [25], [26], [27]).

We have considered the contemporary dependence of $h(\nu)$ on ν in different frequency bands. But we also need the numerical values of $h(\nu)$. They can be derived from the value of h attributed to the longest waves which are presently within the Hubble radius, that is to the waves at frequency $\nu \approx 10^{-18}\ Hz$. These waves did not diminish yet their amplitude adiabatically, so their amplitude is actually determined by its value at the end of inflation. According to the amplification mechanism, this is the same numerical amplitude, as the one these waves had before the beginning of the amplification process. By this logic we arrive at the number

$$h \approx h_i \simeq \frac{H_0}{n} A(n) = \frac{l_{pl}}{\lambda_0},$$

where λ_0 is the wavelength equal to the Hubble distance at the De-Sitter stage: $l_0 = \frac{c}{H_0}$. In principle, the value of H_0 could have been arbitrary. However, if H_0 was too large, such that

$$\frac{l_{pl}}{l_0} > 10^{-4},$$

then it would mean that the value of h ($\nu \approx 10^{-18}\,Hz$), the amplitude of waves at $\nu \approx 10^{-18}\,Hz$, was larger than 10^{-4}. Since the variations of the microwave background temperature, $\frac{\Delta T}{T}$, caused by these waves, are essentially equal to h ($\nu \approx 10^{-18}\,Hz$), this would contradict the existing upper limits on $\frac{\Delta T}{T}$. Thus, h ($\nu \approx 10^{-18}\,Hz$) $\lesssim 10^{-4}$. Similar considerations apply to $\frac{\Delta T}{T}$ caused by waves with frequencies $\nu \approx 10^{-16}\,Hz$ and to experimental limits on $\frac{\Delta T}{T}$ in the corresponding angular scales (several angular degrees). All that is summarized in Fig. 4. This figure shows different theoretical predictions on stochastic gravitational waves, the existing experimental limits and the expected levels of sensitivity from various techniques [20], [28].

Chapter 53

Quantum States of a Harmonic Oscillator

Before going to quantum-mechanical treatment of the graviton creation, let us recall some properties of quantum states of an ordinary harmonic oscillator of any nature — mechanical, electromagnetic, etc. Especially, we will be interested in the notion of squeezed quantum states.

Classical equations of motion for a harmonic oscillator, $\ddot{x} + \omega^2 x = 0$, can be derived from the Lagrange function

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

according to the rule:

$$\dot{\left(\frac{\partial L}{\partial \dot{x}}\right)} - \frac{\partial L}{\partial x} = 0.$$

Associated with L is the Hamiltonian function

$$H = \frac{p^2}{2m} + \frac{m}{2} \omega^2 x^2,$$

where

$$p = \left(\frac{\partial L}{\partial \dot{x}}\right).$$

Quantization is achieved by introducing the operators $\hat{x}$ and $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ and establishing the commutation relation by:

$$[\hat{x}, \hat{p}] = i\hbar.$$

From $\hat{x}$ and $\hat{p}$ one can construct the creation and annihilation operators $a^\dagger$ and a:

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(\hat{x} - i\,\frac{\hat{p}}{m\omega}\right)$$

$$a = \left(\frac{m\omega}{2\hbar}\right)^{2} \left(\hat{x} + i\,\frac{\hat{p}}{m\omega}\right)$$

$$\left[a, a^\dagger\right] = 1$$

and the particle number operator $\hat{N}$:

$$\hat{N} = \hat{a}^\dagger\,\hat{a} = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}.$$

The oscillator can be described by the wave function (or state function) $\psi(x.t)$ which satisfies the Schrödinger equation:

$$i\hbar\,\frac{\partial\psi}{\partial t} = \hat{H}\,\psi.$$

The ground (vacuum) quantum state $|0>$ is defined by the requirement $a|0>= 0$. The ground wave function has the form

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\, exp\left[-\frac{m\omega}{2\hbar}\,x^2\right].$$

The n-quantum states are defined as the eigenstates of the $\hat{N}$ operator:

$$\hat{N}\,|n>= n\,|n>.$$

They are also eigenstates of $\hat{H}$ with eigenvalues $\hbar\omega\left(n+\frac{1}{2}\right)$,

$$\hat{H}\,|n>=\hbar\omega\,|n>.$$

These states are produced by the action of the creation operator $a^\dagger$ on the vacuum state:

$$|n>=\frac{(a^\dagger)^n}{\sqrt{n!}}\,|0>.$$

An important class of quantum states called the coherent states, is generated from the vacuum state $|0>$ by the action of the displacement operator:

$$D(a,\alpha)\equiv exp[\alpha\,a^\dagger-\alpha^*\,a],$$

where α is an arbitrary complex number. Symbolically one can write $|CS>=\hat{D}\,|0>$. The squeezed states (for a review see [29]) are generated by the action of the squeeze operator:

$$S(r,\phi)\equiv exp\left[\frac{1}{2}\,r\,(e^{-2i\phi}\,a^2-e^{2i\phi}\,(a^\dagger)^2)\right],$$

where r and ϕ, known as the squeeze factor and the squeeze angle, are real numbers: $0\le r<\infty$, $-\frac{\pi}{2}<\phi\le\frac{\pi}{2}$. A squeezed state is generated by the action of the squeeze operator on any coherent state and, in particular, on the vacuum state. Symbolically, one can write $|SS>=\hat{S}\,\hat{D}\,|0>$. If a harmonic oscillator is exposed to the time-dependent interaction, then a squeezed state is produced by the Hamiltonian

$$H(t)=\frac{1}{2}\,i\,(\xi^*\,a^2-\xi\,(a^\dagger)^2)\tag{1}$$

where ξ is an arbitrary function of time.

The meaning of the word "squeezed" is related with the properties of these states with respect to variances (or noise moments) ΔA of different operators A:

$$\Delta A\equiv A-<A>.$$

The squeezed wave functions are always Gaussian:

$$\psi(x) \approx e^{-\frac{1}{2}\gamma\, x^2} \tag{2}$$

However, the variances of, say, variables $\hat{x}$ and $\hat{p}$ are substantially different. They can be presented in terms of the complex parameter γ,

$$\gamma \equiv \gamma_1 + i\gamma_2,$$

or real parameters r, ϕ:

$$< (\Delta\hat{x})^2 > \; = \frac{1}{2\gamma_1} \tag{3}$$

$$< (\Delta\hat{p})^2 > \; = \frac{|\gamma|^2}{2\gamma_1} \tag{4}$$

$$< (\Delta\hat{x}\,\Delta\hat{p} >_{sym} \; = -\frac{\gamma_2}{2\gamma_1} \tag{5}$$

$$< (\Delta\hat{x})^2 > \; = \frac{1}{2}\,(\cosh 2r - \sinh 2r\ \cos 2\phi) \tag{6}$$

$$< (\Delta\hat{p})^2 > \; = \frac{1}{2}\,(\cosh 2r + \sinh 2r\ \cos 2\phi) \tag{7}$$

$$< \Delta\hat{x}\,\Delta\hat{p} >_{sym} \; = -\frac{1}{2}\ \sinh 2r\ \sin 2\phi \tag{8}$$

These variances sholud be compared with those for a coherent state, which are always equal to each other and are minimally possible:

$$< (\Delta\hat{x})^2 > \; = < (\Delta\hat{p})^2 >= \frac{1}{2}$$

$$< \Delta\hat{x}\,\Delta\hat{p} >_{sym} \; = 0.$$

So, in a squeezed state, one component of the noise is always "squeezed" with respect to another. In $(x,\, p)$ plane the line of a constant total noise

$$K = \frac{1}{2}\left[< (\Delta\hat{x})^2 > + < (\Delta\hat{p})^2 >\right]$$

for the coherent states can be described by a circle, while this line is an ellipse for the squeezed states (see Fig. 5).

Chapter 54

Squeezed Quantum State of Relic Gravitons. Theory and Experimental Prospects.

Now we return to the cosmological graviton production[1]. The basic equation (3) can be derived from the Lagrange function:

$$L = \frac{1}{2}\frac{M}{a}y'^2 - \frac{1}{2}aM\Omega^2 y^2,$$

where $y \equiv \mu/a$, $M \equiv a^3$, $\Omega \equiv n/a$. The Hamiltonian $\hat{H}$ of the system constructed in terms of coordinate $\hat{y}$ and momentum $\hat{P} = -i\frac{\partial}{\partial y}$ operators has the form

$$\hat{H} = \frac{\hat{P}_y^2}{2M} + \frac{1}{2}M\Omega^2 y^2,$$

so that the Schrödinger t-time equation ($dt = ad\eta$) is[2]

$$i\frac{\partial\psi}{a\partial\eta} = \hat{H}\psi. \tag{1}$$

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[1]Below we will mainly follow our recent paper [20]

[2]Here and below we set $\hbar = 1$.

Let us rewrite $\hat{H}$ in terms of creation $\hat{b}^+$ and anhilitation $\hat{b}$ operators wich make the Hamiltonian diagonal at any initial time $\eta = \eta_b$:

$$\hat{b}^+ = \left(\frac{M_b\Omega_b}{2}\right)^{1/2}\left(\hat{y} - i\frac{\hat{p}}{M_b\Omega_b}\right),$$

$$\hat{b} = \left(\frac{M_b\Omega_b}{2}\right)^{1/2}\left(\hat{y} + i\frac{\hat{p}}{M_b\Omega_b}\right), \tag{2}$$

$$\hat{H} = f_1(a)\left(\hat{b}^+\hat{b} + \hat{b}\hat{b}^+\right) + \tag{3}$$

$$+ f_2(a)\left(\hat{b}^{+^2} + \hat{b}^2\right) \tag{4}$$

where $f_1 = \frac{\omega_b}{4M}(1 + \frac{\omega^2}{\omega_b^2})$, $f_2 = \frac{\omega_b}{4M}(\frac{\omega^2}{\omega_b^2} - 1)$, $\omega = M\Omega$, $\omega_b = M_b\Omega_b$. The functions f_1 and f_2 depend on the scale factor a, that is they depend on time. The first term in 4 belongs to the quantum number conserving interactions, while the second term, with the coeficient $f_2(a)$, have the form of non-conserving interaction Hamiltonian 1 producing squeezed quantum states.

We will seek for the solution to eq. (8) in the general form

$$\psi = C(\eta)e^{-B(\eta)y^2} \tag{5}$$

For $\eta = \eta_b$ and under condition $B(\eta_b) = \frac{1}{2}\omega_b$ the wave function (5) describes the vacuum state. At any subsequent moment of time η the wave function (5) describes a squeezed quantum state. By substituing (5) into eq. (1) one obtains the eqs.:

$$B' = i\left(\frac{1}{2}M\Omega^2 a - 2\frac{B^2}{M}a\right) \tag{6}$$

$$\frac{C'}{C} = -iB\frac{a}{M} \tag{7}$$

The prefactor $C(\eta)$ is determined from eq. (7) $B(\eta)$. As for eq. (6), it is equivalent to eq. (6) if the replacement $B(\eta) = -\frac{i}{2}a^2\frac{(\mu/a)'}{\mu/a}$ is made. Thus, by studying the classical solutions to eq. (3) one can find $B(\eta)$ and, eventually, the squeeze and phase parameters of the squeezed state at any subsequent time, including the present epoch, $\eta = \eta_0$.

It can be shown from eq. (2, 5 and 5) thet $B(\eta)$ is related with $r(\eta)$ and $\varphi(\eta)$ by the formula

$$B = \frac{\omega}{2}\frac{\cosh r + e^{2i\varphi}\sinh r}{\cosh r - e^{2i\varphi}\sinh r}$$

so that $r = \frac{1}{2}Arccosh\left(\frac{\omega^2+4B^*B}{4\omega ReB}\right)$, $sin 2\varphi = \frac{1}{\sinh 2r}\frac{ImB}{ReB}$. The mean value of quanta in a given mode is equal to

$$< N >= \frac{(\omega - 2ReB)^2 + 4(ImB)^2}{8\omega ReB}$$

so that one can also find out r from the relation $r = Arcsinh(< N >)^{1/2}$.

A concrete cosmological model wich we already discussed above included three successive stages of expansion: inflationary (i) governed by the effective cosmological constant Λ ($\Lambda <_{\sim} \Lambda_{Pl}$), radiation-dominated (e) governed by matter with the equation of state $p = \epsilon/3$ and matter dominatted (m) governed by matter with the equation of state $p = 0$. The scale factors at these three stages are:

$$\begin{aligned} a_i &= -(H_0\eta)^{-1}\,(-\infty < \eta \leq \eta_1), \\ a_e &= c_1(\eta + \theta_1)\,(\eta_1 \leq \eta \leq \eta_2), \\ a_m &= c_2(\eta + \theta_2)^2\,(\eta \geq \eta_2), \end{aligned}$$

where it is assumed that $a(\eta)$ and $a'(\eta)$ are joined continuously at η_1 and η_2. From the joining conditions we obtain, in particular, $\theta_1 = -2\eta_1$, $\theta_2 = \eta_2 - 4\eta_1$. At the three stages the basic solutions ξ to eq. (3) have the form:

$$\begin{aligned} \xi_i &= \left(1 - \frac{i}{n\eta}\right) e^{-in\eta}, \\ \xi_e &= e^{-in(\eta+\theta^1)}, \\ \xi_m &= \left(1 - \frac{i}{n(\eta+\theta_2)}\right) e^{-in(\eta+\theta_2)}. \end{aligned} \tag{8}$$

The general solution to eq. 3 is an arbitrary linear combination of ξ and ξ^*: $\mu = delta^{(1)}\xi + \delta^{(2)}\xi^*$. The continuous joining of solutions $\xi_i = \alpha_1\xi_e + \beta_1\xi_e^*, \xi_e = \alpha_2\xi_m +$

$\beta_2\xi_m^*$ allows us to derive the relation:

$$\xi_i = (\alpha_1\alpha_2 + \beta_1\beta_2^*)\xi_m + (\alpha_1\beta_2 + \beta_1\alpha_2^*)\xi_m^*,$$

where

$$\begin{aligned}
\alpha_1 &= \left(1 - \frac{i}{n\eta_1} - \frac{1}{2n^2\eta_1^2}\right) e^{-2in\eta_1}, \\
\beta_1 &= \frac{1}{2n^2\eta_1^2}, \\
\alpha_2 &= \left(1 + \frac{i}{n\theta} - \frac{1}{2n^2\theta^2}\right) e^{+in\theta/2}, \\
\beta_2 &= -\frac{1}{2n^2\theta^2} e^{-3in\theta/2}, \\
\theta &\equiv 2(\theta_2 - \theta_1).
\end{aligned}$$

Now let us compute the value of the squeezing factor for a given mode n: $r(n)$. We will consider all waves shorter than the present Hubble radius: $|n(\eta_0 + \theta_2)| \gg 1$.

By assumption, at the inflationary stage the initial quantum state for gravitons was vacuum, that is $B(\eta_b) = \omega(\eta_b)/2 = na^2(\eta_b)/2$ at $\eta = \eta_b$. From this assumption one can determine δ_i — the value of $\delta \equiv \delta^{(1)}/\delta^{(2)}$ at the inflationary stage. One obtains $\delta_i = (1 + 2in\eta_b)^{-1} exp(2in\eta_b)$, that is δ_i is vanishingly small for $|n\eta_b| \gg 1$ (for high frequency modes at $\eta = \eta_b$). From the joining conditions one derives the relationship between δ_i and δ_m, i.e. δ at i- and m- stages:

$$\delta_m = \frac{(\alpha_1^*\beta_2^* + \beta_1^*\alpha_2) + \delta_i(\alpha_1\alpha_2 + \beta_1\beta_2^*)}{(\alpha_1^*\alpha_2^* + \beta_1^*\beta_2) + \delta_i(\alpha_1\beta_2 + \beta_1\alpha_2^*)}$$

It is sufficient to know only δ_m in order to calculate $r(n)$ for the present epoch. Indeed, it can be shown that $< N >= \delta_m^*\delta_m(1 - \delta_m^*\delta_m)^{-1}$ and therefore: $r \approx Arsinh\,[\delta_m^*\delta_m(1 - \delta_m^*\delta_m)^{-1}]^{-1/2}$.

Let us first find $r(n)$ in the frequency range $|n\eta_1| \ll 1, |n\theta| \gg 1$, that is for waves wich have been amplified during i-stage only. Their contemporary frquencies are larger that $10^{-16} Hz$. In this range one has:

$$\alpha_1 = -(2n^2\eta_1^2)^{-1},$$
$$\beta_1 \approx (2m^2\eta_1^2)^{-1},$$
$$\alpha_2 = exp(in\theta/2),$$
$$\beta_2 = 0. \tag{9}$$

Since $\sinh r \approx e^r/2$ for $r \gg 1$ we obtain the following result:

$$r \approx \ln \frac{a_2(n)}{a_1(n)} \tag{10}$$

where a_1 is the value of the scale factor at the time of exit of a given mode out of the Hubble radius at i- stage and a_2 is the value of a at time of returning of the mode to under the Hubble radius at e-stage. In orde to get the numerical values of $r(n)$ one first observes that r goes to zero for waves approaching $|m\eta_1| \approx 1$ and stays zero for $|n\eta_1| \gg 1$ that is for modes wich always remain in the vacuum state. The contemporary frequency of waves corresponding to $|n\eta_1| \sim 1$ is $\nu \approx 10^8 Hz$ (for $l_0 \sim 10^4 l_{Pl}$, where l_0 is the Hubble radius at i-stage). For lower contemporary frequencies the squeezing parameter $r(n)$ increases according to eq. (10) and reaches the value $r \approx \ln 10^{48} \approx 10^2$ at $\nu \approx 10^{-16} Hz$.

Now let us turn to the region $|n\eta_1|, |n\theta| \ll 1$, that is to waves with contemporary frequencies in the range $10^{-18}Hz < \nu < 10^{-16}Hz$. These waves have been amplified during both i- and m- stages. In this case we have α_1, β_1 given in eq. (9) and α_2, β_2 in the form :

$$\alpha_2 \approx -\frac{1}{2n^2\theta^2} + \frac{3i}{4n\theta}, \beta_2 \approx -\frac{1}{2n^2\theta^2} + \frac{3i}{4n\theta}.$$

For r we have the result:

$$r \approx \ln \frac{1}{n\theta n^2 \eta_1^2} \approx \ln \frac{a_2(n)}{a_1(n)} \tag{11}$$

where a_1 is the same as in eq. (10) and a_2 correspons to time when the wave returns under the Hubble radius at m- stage. The numerical value of r changes in accord with eq. (11) from $r \approx 10^2$ at frequency $\nu \approx 10^{-16} Hz$ uo to $r \approx \ln 10^{48} 10^6 \approx 1.2\, 10^2$ at $\nu \approx 10^{-18} Hz$.

Thus we see that the mode wave function ψ_{nlm} for $h_{nlm} = \frac{1}{a}\mu(\eta)G_{nlm}(x)$ describes a squeezed state with the squeezing coeficient $r(n)$ independent of 1 and m. The total ave function describing the relic gravitational wave background is the product of the single-mode squeezed wave functions: $\psi = \overbrace{\{n,l,m\}}\psi_{nlm}$.

The graviton statistics in the considered state generated by squeezong the vacuum is super-Poissonian. In the possible gravitational wave experiments similar to those well known in optics, the squeezed states discussed here would manifest themselves in the form of graviton superbunching. Another way of observing the predicted squeezing is the analysis of noises in $sin\omega t$ and $cos\omega t$ components of the gravitational-wave antenna output.

It is worth noting that similar statements about final squeezed quantum sate apply also to other possible zero-point fluctuations amplified during the inflationary stage.

Thus, we see that the relic gravitational waves should exist now in the squeezed quantum state whose parameters can, in principle, be determined experimentally.

Chapter 55

Quantum Cosmology, Minisuperspace Models and Inflation

Until now we have been discussing the quantum fluctuations superimposed on a given classical background spacetime. This is the right time now to start discussing the quantization of the background geometry itself. In other words, we have reached the domain of quantum cosmology.

In canonical quantum gravity the role of a generalized coordinate is played by a 3-geometry $g^{(3)}$. The full set of 3-geometries forms a superspace, where the wave function of the Universe is defined. If some matter fields are present as well, the superspace includes the variables describing the values of the matter fields on 3-geometries. The basic equation which governs the wave function of the Universe is known as the Wheeler-DeWitt (WD) equation. (For reviews of quantum gravity and quantum cosmology, see, for example, [31], [32], [33], [34].)

A simplified case, which allows a detailed investigation, is provided by minisuperspace models. In minisuperspace models one neglects all degrees of freedom except a few. Reasonably simple, though sufficiently representative, is a quantum cosmological model describing a homogeneous isotropic universe filed with a massive scalar field. In this case one has only two degrees of freedom (two minisuperspace variables) the scale factor $a(t)$ and the scalar field $\phi(t)$. Since the transition to the notions of quantum gravity includes the integration of some quantitites such as the Hamiltonian function over the 3-volume, one normally considers closed 3-geometries, $k = +1$, in order to avoid infinite expressions. The total energy of a closed world is zero, that is why the analog of the Schrödinger equation takes the form $\hat{H}\psi = 0$, which is the Wheeler-DeWitt equation.

For the case of a FRW universe with the scalar field ϕ, $V(\phi) = \frac{1}{2} m^2 \phi^2$, the Wheeler-DeWitt equation can be written as follows [35], [36]:

$$\left(\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial}{\partial\phi^2} - ka^2 + m^2\phi^2a^4\right)\psi(a,\phi) = 0 \tag{1}$$

The factor p reflects some ambiguity in the choice of operator ordering. The possible preferred choice pf p for the given model is $p = 1$.

First, we will show how classical Einstein equations of motion follow from the quantum equation (1) in the quasi- classical limit. For simplicity we consider the limit where the spatial curvature term, ka^2, can be neglected. In this limit (and for $p = 1$) eq. (1) reduces to

$$\left(\frac{1}{a}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} + m^2\phi^2a^4\right)\psi(a,\phi) = 0 \tag{2}$$

In the quasi-clasical approximation, the wave function $\psi(a,\phi)$ is of the form

$$(\hbar = 1): \ \psi(a,\phi) = exp(iS(a,\phi) + i\sigma(a,\phi) + ...).$$

By using this representation the following equations can be derived from eq. (2):

$$-\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{a^2}\left(\frac{\partial S}{\partial\phi}\right)^2 + m^2\phi^2a^4 = 0 \tag{3}$$

$$i\frac{\partial^2 S}{\partial a^2} - 2\frac{\partial S}{\partial a}\frac{\partial\sigma}{\partial a} + \frac{i}{a}\frac{\partial S}{\partial a} - \frac{i}{a^2}\frac{\partial^2 S}{\partial\phi^2} + \frac{2}{a^2}\frac{\partial S}{\partial\phi}\frac{\partial\sigma}{\partial\phi} = 0 \tag{4}$$

Eq. (3) is the Hamiltonian-Jacobi equation for the action S. Areal solution of (3), which describes the classical dynamics of the model, can be represented in the form

$$S(a,\phi) = -a^3 f(\phi) \tag{5}$$

An unimportant additive constant has been omitted here. From (3) it follows that the function $f(\phi)$ satisfies the ordinary differential equation

$$9f^2 - \left(\frac{df}{d\phi}\right)^2 = m^2\phi^2 \tag{6}$$

The classical equations of motion are obtained from (5) and the system Lagrangian

$$L = \frac{1}{2}\left(-a\dot{a}^2 + a^3\dot{\phi}^2 - a^3 m^2\phi^2\right)$$

in the usual way:

$$\frac{\partial L}{\partial \dot{a}} = -a\dot{a} = \frac{\partial S}{\partial a}$$

$$\frac{\partial L}{\partial \dot{\phi}} = a^3\dot{\phi} = \frac{\partial S}{\partial \phi}.$$

This then gives the relations

$$\begin{aligned} \frac{\dot{a}}{a} &= 3f \\ \dot{\phi} &= -f' \end{aligned} \tag{7}$$

The prime here signifies differentiation with respect to Φ, and the dot denotes differentiation with respect to time. Differentiating (7) with respect to t and using (6), we can obtain the equations of motion in the usual classical form:

$$\begin{aligned} &\ddot{\phi} + 3\frac{\dot{a}}{a}\,\dot{\phi} + m^2\phi = 0 \\ &\left(\frac{\dot{a}}{a}\right)^2 = \dot{\phi}^2 + m^2\phi^2 \\ &\left(\dot{\frac{\dot{a}}{a}}\right) + \left(\frac{\dot{a}}{a}\right)^2 = -2\dot{\phi}^2 + m^2\phi^2 \end{aligned} \tag{8}$$

These equations of motion are invariant under the transformation $t \to -t$. The three equations of motion can be combined to give one, in which the time parameter t does not appear:

$$\phi \frac{d^2\phi}{d\alpha^2} + \left(3\phi \frac{d\phi}{d\alpha} + 1\right) \left[1 - \left(\frac{d\phi}{d\alpha}\right)^2\right] = 0.$$

(For the sake of convenience, we use the variable $\alpha = \ln a$ from here on.) This equation completely describes the classical trajectories. The direction of motion is determined by the choice of direction in time.

At this point we should clarify the issue of whether the sign of the action S has anything to do with such notions as expansion or contraction of a cosmological model. This issue is discussed in the context of "tunelling" wave functions [37], [38]. Eq. (5) and (7) may lead to the impression that $S < 0$ (and $f > 0$) corresponds to the expansion ($\dot{a} > 0$), while the opposite choice $S < 0$ ($f < 0$) corresponds to the contraction ($\dot{a} < 0$) of the cosmological volume. However, the choice of the parameter t in these equations is absolutely arbitrary. The functions $S > 0$ ($f < 0$) can perfectly describe expansion if one just changes the parameter t to $-t$ in eq. (7). Thus, the sign of the action does not attach a definite meaning to the direction of evolution along the classical trajectories.

All trajectories of the model (8) in the $(\phi, \dot{\phi})$ phase plane have previously been found [39] and it has been shown that in the case of expansion (i.e., $\dot{a} > 0$), the trajectories all start out from two ejecting nodes K_1 and K_2 (see Fig. 6). Apart from these trajectories, there are also two attracting separatrices that originate at two saddle points, S_1, S_2. The solutions of eq. (6) have the following asymptotic behaviour for trajectories that start out from the nodes:

$$f \approx c\, e^{\pm 3\phi}, \qquad c^2\, e^{\pm 3\phi} \gg m^2 \phi^2, \qquad c = const.$$

and for the separatrices

$$f \approx \pm \frac{1}{3}\, m\phi, \qquad 9\phi^2 \gg 1.$$

Different values of C select different trajectories leaving the nodes. Consequently, the choice of a definite solution of eq. (6) gives a definite function S and a definite classical trajectory.

We will distinguish different solutions of eq. (6) by the subscript n, which varies continually and takes on two values, corresponding to the separatrices. By virtue of the linearity of the WD equation, we can symbolically write a more general solution of eq. (2) to lowest order in the form

$$\psi = \sum_n exp(iA_n + iS_n), \quad S_n = -exp(3\alpha)f_n, \quad A_n = const.$$

To every quasi-classical wavefunction $\psi_n = exp(iS_n)$ one can put into correspondence a family of normals to the surfaces $S_n = const$. (Fig. 7) These surfaces are constructed in the minisupespace $(a,\ \Phi)$, with metric tensor

$$G^{\mu\nu} = e^{-3\alpha} diag(-1, +1), \quad \mu, \nu = 1, 2, \quad x^1 = \alpha,\ x^2 = \phi \tag{9}$$

The vector normal N_μ to $S_n = const$ can be obtained by acting on $\psi_n = exp(iS_n)$ with the momentum operators π_α and π_ϕ:

$$\hat{\pi}_\alpha\, \psi_n = \frac{1}{i} \frac{\partial}{\partial \alpha}\, \psi_n = \frac{\partial S_n}{\partial \alpha}\, \psi_n = N_\alpha\, \psi_n$$

$$\hat{\pi}_\phi\, \psi_n = \frac{1}{i} \frac{\partial}{\partial \phi}\, \psi_n = \frac{\partial S_n}{\partial \phi}\, \psi_n = N_\phi\, \psi_n$$

Taking into account eqs. (5), (9) one obtains $N^\alpha = 3f$, $N^\phi = f'$. Integrating the relation

$$\frac{d\alpha}{d\phi} = \frac{N^\alpha}{N^\phi} = -\frac{3f}{f'}$$

along every classical path in the $(\alpha,\ \phi)$ plane, one gets $z(\alpha,\ \phi) = const$, where

$$z \equiv \alpha + 3\int \left(\frac{f}{f'}\right) d\phi.$$

In the case at hand the family of normals to S_n and associated tangent vectors N^α, N^ϕ are independent of α and transform back into themselves under the shift $\alpha \to \alpha + const$, or $a(t) \to const \bullet a(t)$. This symmetry is the reflection of the fact that the function $a(t)$ alone does not appear in eq. (8). Since the vector $(N^\alpha$,

N^ϕ) points in the same direction as the momentum vector (Π^α, Π^ϕ), the normals trace out classical trajectories in (α, ϕ) space. Therefore, invariance of the family of normals under the displacement $\alpha \to \alpha + const$ signifies that for a given S_n, the curves traced out by normals are all copies of the same classical solution in the (ϕ, $\dot{\phi}$) plane.

Thus, we learned to see that different solutions to the Hamilton-Jacobi equation determine different wave functions in their lowest (in terms of $\hbar$) approximation. On the other hand, to a given S_n one can assign a family of classical trajectories. In the case considered above they all happened to be the copies of one and the same physically distinct classical solution. However, it is just a consequence of our flat-space approximation, $k = 0$. In general, it is not the case. One can analyse [40], for example, the family of classical solutions singled out by one particular wavefunction, namely by the Hartle-Hawking wavefunction.

The next $\hbar$-order approximation to S_n defines the prefactor to the wave function $\psi_n \sim e^{iS_n}$. The prefactor is responsible for forming a packet from classical trajectories determined by S_n. It assigns different "weights" to different paths orthogonal to $S_n = const$.

Let us return to eq. (4) for σ_n. The general solution for σ_n can be expressed in terms of the function $f_n(\phi)$:

$$\sigma_n(\alpha, \phi) = \frac{i}{2}\left(3\alpha + \ln f_n'\right) + B_n(z),$$

where B_n is an arbitrary function of its argument z. In the present approximation, the general solution to WD equation can be written in the form

$$\psi = \sum_n Z_n\, \psi_n = \sum_n Z_n\, exp(iS_n + i\sigma_n) \tag{10}$$

where

$$\psi_n = \xi_n\, e^{i\gamma_n} \left(a^3 f_n'\right)^{-1/2} e^{-ia^3 f_n}. \tag{11}$$

ξ_n and γ_n are arbitrary real functions of z, Z_n are arbitrary complex numbers. One can see that to every path $z = const$ in (α, ϕ) plane one can assign the number $Q_n \equiv \xi_n^2(z)$ which is conserved along this path. The particular value of Q_n is determined by the chosen boundary conditions for the wave function, and specifically by the function $\xi_n(z)$.

Chapter 56

Form the Space of Classical Solutions to the Space of Wave Functions

From the problems of distributing "weights "ampng different classical trajectories belonging to the same family determined by S_n we now turn to the more difficult problem of distributing "weights "among the wavefinctions themselves. As we saw above. the WKB components ψ_n ,(11), participate in the general solution (10) with arbitrary complex coefficients Z_n. They determine one or other choice of possible wavefunctions. How can one classify the space of all possible wavefunctions?

To answer this question we will start from the simplest situation, when the number of linearly independent solutions to WD equationis just two. For this aim we will first consider eq. (1) in another limiting case, namely, when the term $-\frac{1}{a^2}\frac{\partial}{\partial\phi^2}$ can be neglected. In this case the variable ϕ plays a role of a parameter and the problem reduces to the one-dimensional problem. The basic eq. (1) can be written in the form (for $k = +1$):

$$\left(\frac{1}{a^p}\frac{d}{da}a^p\frac{d}{da} - a^2 + H^2a^4\right)\psi(a) = 0 \tag{1}$$

where $H \equiv m^2\phi^2$. We prefer to work with exact solutions to eq. (1) so we choose $p = -1$ or $p = 3$ [41] [42][1]. We will write the exact solution for the $p = -1$ case

[1]The case $p = -1$ was first considered in ref. [43].

$$\psi(a) = u^{1/2}\left[A_1 H^{(1)}_{1/3}\left(\frac{u^{3/2}}{3H^2}\right) + A_2 H^{(2)}_{1/3}\left(\frac{u^{3/2}}{3H^2}\right)\right], \ H^2a^2 \geq 1, \quad (2)$$

$$\psi(a) = (-u)^{1/2}\left[B_1 I_{1/3}\left(\frac{(-u)^{3/2}}{3H^2}\right) + B_2 K_{1/3}\left(\frac{(-u)^{3/2}}{3H^2}\right)\right], \ H^2a^2 \leq 1, \quad (3)$$

where $u = H^2a^2 - 1$ and $I_{1/3}, K_{1/3}, H^{(1)}_{1/3}, H^{(2)}_{1/3}$ are the Infield, Macdonald and Hankel special functions.

Eq. (1) has the form of the Schrödinger equation for a 1-dimensional problem with the potential $V(a) = a^2 - H^2a^4$ (see Fig. 8). A_1, A_2 and B_1, B_2 are two pairs of constant arbitrary coefficients multiplying two pairs of linearly independent solutions. By matching the solutions (2) and (3) at the point $a = 1/H$ one finds [41], [42]:

$$B_1 = -A_1(1 + i\sqrt{3}) - A_2(1 - i\sqrt{3}), B_2 = \frac{2i}{\pi}(A_2 - A_1).$$

Now let us characterize the full space of the wave functions[2]. In the present case, our quantum system has just two linearly independent states and therefore resembles a simple spin $-1/2$ system. Let us call arbitrarily chosen basis of states $|1>$ and $|2>$. A general state $|\psi>$ can be expanded as $|\psi> Z_1|1> + Z_2|2>$, where Z_1 and Z_2 are arbitrary complex constants.

It is a general principle of quantum mechanics that state vectors wich differ only by an oerall non-zero multiple λ describe one and the same physical state. Thus, the pair of coordinates (Z_1 and Z_2) and the pair (λZ_1 and λZ_2) are equivalent. It follows that physical quantities can only depend on the ratio $\zeta = Z_1/Z_2$ wich is invariant under rescaling. In our example above we may identify Z_1 with B_1 and Z_2 with B_2. It is convenient to introduce the notation $B_1 = |B_1|exp(i\beta_1)$, $B_2 = |B_2|exp(i\beta_2)$, $\beta = \beta_1 - \beta_2$ and then $\zeta = \frac{B_1}{B_2} = xexp(i\beta)$. The ratio ζ parametrizes the points on a 2-dimensional sphere and so we see that the set of possible wavefunctions is in $1-1$ correspondence with the points on the 2-sphere.

We now wish to place a measure on the space of quantum states. Of course there are many possible measures. However, in choosing a measure we should

[2]Here we mainly follow to ref. [44].

be guided by the principle that the measure should be independent of the arbitrary choice of basis states $|1>$ and $|2>$. That is if we perform a unitary change of basis, wich will preserve all probability amplitudes, then the measure should remain invariant.

The invariance of the measure may be taken as the quantum analogue of the principal of general covariance in classical general relativity. In fact in the classical limit it corresponds to invariance under canonical transformations. This latter invariance was used in ref. [45] to suggest a suitable measure on the set of classical solutions.

For a 2-state system the 2-dimensional unitary transformations will act (provided $|1>$ and $|2>$) are normalised) on the complex 2-vector (Z_1 and Z_2) by multiplication by a 2 by 2 unitary matrix. Clearly the ratio ζ is unaffected by matrices wich are merely multiples of the unitary matrix so we may confine attention to special unitary matrices of determinant unity, this still allows minus the identity matrix so if we want just the transformations wich change the physical states we must identify to $SU(2)$ matrices wich differ by multiplication by minnus one. That is, the effective physical transformations acting on the space of quantum states is the rotation group $SO(3) = SU(2)/C_2$ where C_2 is the group consisting of $+1$ and -1. In fact this acts on the 2-sphere in the usual way provided we identify β with the longitudinal angle and $x = cotan(\theta/2)$ where θ is the usual co-latitude.

It is now clear that we mut choose for our invariant measure on the space of quantum states the usual volume element on the 2-sphere. This is clearly invariant under rotations and up to an arbitrary constant multiple it is unique. That is the measure on terms of β θ:

$$dV = sin\theta\, d\theta\, d\beta,\ 0 \le \theta \le \pi,\ 0 \le \beta \le 2\pi \tag{4}$$

Of course the measure is just the riemannian volume element with respect to the standard round metric on the 2-sphere.

It should be mentioned that the well known Hartle-Hawking wavefunction [46] is exactly the south pole ($\theta = \pi$) of the 2-sphere. This wavefunction is real. Another real wavefunction corresponds to the north pole of the 2-sphere. We call this wave function anti-Hartle-Hawking wavefunction. All other wavefunctions are complex.

Chapter 57

On the Probability of Quantum Tunneling from "Nothing "

The measure introduced in the space of all wavefunctions may allow us to formulate and solve physically meaningful problems. We will try to pose one such a problem already in the considered simplest model. As was mentioned above, eq. (1) looks like the Schrödinger equation for a particle moving in the presence of the potential $V(a)$. The form of the potential (Fig. 8) motivates the expectation that some of the wave-functions may be capable of describing the quantum tunneling or decay. In ordinary quantum mechanics the quantity D, where $D = \frac{\psi(a_2)}{\psi(a_1)}|^2$ can be interpreted as the quasiclassical probability for the particle to tunnel from one classicaly allowed region to another (see Fig. 8). The wave- function used in this expression is determined by the imposed boundary conditions, i.e. it is determined by the physical formulation of the problem. The value of D is always (much) less than unity, $D < 1$, for wave-functions describing quantum tunneling or decay. One can define a similar quantity D in our quantum cosmological model. however, the physical interpretation of D is not clear. The main difference from the quantum mechanics is that in ordinary quantum mechanics one imposes suitable boundary conditions in time t abd space x, while in our problem there is only one coordinate, a[1] . Nevertheless, we will adopt the same definition of D in our problem: $D = |\frac{\psi(\frac{1}{H})}{\psi(0)}|^2$, $p < 1$. This quantity is well defined mathematically and can be calculated for every solution of eq. (1)

[1]The notion of the break of classical evolution in quantum cosmology is rather intrincate. We have argued in ref. [47] that only in superspaces of more than one dimension this notation can be clearly formulated.

regardless of its interpretation. Since in our actual problem the "energy "$\epsilon = 0$, we can provisionally interpret D as the probability for the creation of the Universe from "nothing "[2]. Therefore, we are interested to know wich wave-functions predict $D < 1$ and wich predict $D > 1$. It is not excluded that the wave-functions predicting $D < 1$ can be eventually justified ti be describing the quantum tunneling from "nothing ", or rather from the "vacuum "of some more deep quantum theory.

It is easy to claculate D in the approximation $H \ll 1$ [41] [42]. One can see that different choices of wave-functions give different values of D. For instance, the Hartle-Hawking wave-function corresponds to the choice $B_1 = 0$ and gives $D = exp(\frac{2}{3H^2} \geq 1$. We are interested in the value of D for a typical wave-function. In other words, how many wave-functions give $D < 1$ or $D > 1$? To answer this question one must consider the space of all possible wave-functions with a suitable measure. By using the measure (4) one can show that the set of wave functions predicting $D > 1$ is very small compared with that predicting $D < 1$. This follows from the fact that the surface area covered by the wave-functions with $D > 1$ is very small compared with the total surface area of the 2-sphere. Indeed, the line separating $D > 1$ and $D < 1$ regions on the 2-sphere corresponds to $\theta_0 \approx \pi - 2exp(-\frac{1}{3H^2})$, $\pi - \theta_0 \ll 1$. Only a small area around the south pole $\theta = \pi$ gives wave-functions with $D > 1$, the rest of the surface of the 2-sphere corresponds to wave-functions with $D < 1$. The ratio of the surface area around the south pole to the total surface area is very small; it is equal to $exp(-\frac{2}{3H^2}) \ll 1$. Thus, one can say, that the probability of finding a wave-function with $D > 1$ (among them is the Hartle-Hawking wave-function) is very small. One can conclude that the overhelminh majority of the wave-functions seem to be capable of descrcibing the quantum tinneling or decay, since they predict $D < 1$[3].

The simple example presented above clarifies the notion of the measure in the space of all physically distinct wave-functions. In the similar way one can introduce the measure in the multi-dimensional space of wave-functions (10) [44]. The use of this measure shows that the inflation is indeed a property of a typical wave-function, at least, under some additional assumptions adopted in [44] [41] [42]. The possibility of assigning different "weight "to different wave-functions suggests that they may be governed by a Wave Function given in the space of all wave-functions, that is the complex coefficients Z_n in eq. (10) may become operators.

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[2]This is slightly more precise formulation of the notion, introduced at the begining of these lectures and graphically depicted in Fig. 1.

[3]Interestingly enough, the product of surface areas with $D > 1$ and $D < 1$ to their corresponding maximal values of D gives approximately equal numbers, both of order of unity.

This is why it was conjectured in [41] [42] that we may find "the concept "of a "secondary "wave-function in the space of all possible wave-functions to be a useful one. It sems to me that the so-called "third "quantization of gravity, so popular now, is an effort in the same direction.

Chapter 58

Relic Gravitons and the Birth of the Universe

The quantum cosmological mini-superspace models analyzed above included only two degrees of freedom and corresponded to homegeneous isotropic universes. The inclusion of all degrees of freedom at the equal footing would present a formidable problem. However, this problem can be simplified in a perturbative approximation. This can be considered as a quantum-mechanical description of a slightly perturbed homogeneous isotropic universe. In particular, the Schrödinger equation for gravitons, eq. (1), can be derived from the fully quantum cosmological approach as some approximate equation.

Let us consider a closed universe governed by an effective cosmological term Λ and perturbed by weak gravitational waves. The WD equation for the wave function of the universe can be written in the form (see, for example, [48]):

$$\left[\frac{1}{2a}\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial}\partial a-\frac{a}{2}+\frac{2}{3\pi}\left(\frac{l_{Pl}}{l_0}\right)^2\frac{a^3}{2}+\sum_{nlm}H_{nlm}(a,h_{nlm})\right]\psi(a,\{h_{nlm}\})=0 \quad (1)$$

Here h_{nlm} denotes the amplitude of the gravity wave perturbation in the mode (n, l, m)[1]. H_{nlm} denotes the Hamiltonian of the perturbations:

$$H_{nlm} = \frac{\pi^2_{nlm}}{2M} + \frac{1}{2} M \Omega_n^2 h^2_{nlm}$$

where π^2_{nlm} is the momentum canonically conjugate to h_{nlm} and $M = a^3$, $\Omega = a^{-1}(n^2 - 1)^{1/2}$. In what follows we will not write indices n, l, m unless they are necessary.

The cosmological wave function depends on a scale factor and a set of h_{nlm} superspace variables: $\psi = \psi(a, \{h\})$. We will assume that $\psi(a, \{h\})$ satisfies the quasiclassical approxiamtion with respect to the variable a. Then the $\psi(a, \{h\})$ can be presented in the form $\psi(a, \{h\}) = exp[-A(a) - A_1(a)]\Phi(a, \{h\})$ where $A(a)$ is the "unperturbed "(background) action and $A_1(a)$ is the prefactor of the background wave function. $\Phi(a, \{h\})$ is hte part of the total wave function describing the fluctuations. We assume that the fluctuations do not affect the background so that the term $\frac{\partial^2 \Phi}{\partial a^2}$ can be neglected in eq. (1).

The wave function $\psi(a, h)$ for each mode of fluctuations obeys the Schrödinger equation

$$-\frac{1}{i}\frac{\partial \psi}{a \partial \eta} = H\psi \tag{2}$$

where $\frac{\partial}{a\partial \eta} = -i\frac{1}{a}\frac{dA}{da}\frac{\partial}{\partial a}$, $\Phi = \prod_{nlm} \psi$. We can see that in the regime when $A(a)$ describes classical Lorentzian evolution (that is, it describes the De-Sitter solution) eq. (2) coincides with eq. (1)[2]. However, eq. (2) can domore than that. It is still valid in the regions where $A(a)$ describes a classically forbidden behaviour of the universe, i.e. in the under-barrier region $a < 1/H$ (see Fig. 8). In this region eq. (2) takes on the form of the Schrödinger equation written in the imaginary time. Thus, the graviton wave function $\psi(a, h)$ extends to the classically forbidden region $a < 1/H$ and is sentitive to the form of the background wave function in this region.

Our final goal is to show that the quantum state of gravitons at the De-Sitter stage (before the amplification has started) may depend on the actual form of

[1]Since we are working in a closed 3-space, it is convenient to attribute he indices l, m to spherical harmonics.

[2]One has to take into account some obvious modifications related to the fact that $k = 0$ was used in eq. (1) and $k = +1$ is used in eq. (2).

the background wave function of the Universe in the region $a < 1/H$. Everywhere in our previous discussion we were assuming that the initial state of gravitons at $\eta = \eta_b$, $\eta_b \to -\infty$ was the vacuum. The present-day observational predicts were also derived under this assumption. However, this assumption, though quite usual and natural, is not obligatory. If the initial state of gravitons could have been a non-vacuum state, then it would lead to to the differing predictions on the to-day spectrum of relic gravitons and their squeeze parameter r. In this way, by measuring the actual parameters of the relic gravitons, one could even learn something about the wave function of the Universe in its classically forbidden regime.

The possible deviations of the initial quantum state of gravitons from the vacuum state can not be too large, they should satisfy two requirements. First, they should not violate our basic assumption that the back-action of gravitons on the background geometry is always negligibly small. Second, they should not lead to predictions for the present day amplitudes which would exceed the existing experimental limits. By combining these requirements one can show that only for low-frequency waves and only for cosmological models with minimally sufficient duration of inflation the initial quantum state of graviton modes can deviate from the vacuum. In this case the deviations of the present-day spectrum can be as large as is shown by a broken line in Fig. 9 for a specific model where $l_0 = 10^9 l_{Pl}$. In this figure the dotted line shows the spectrum produced from the initial vacuum state in the same model. And the solid line shows the highest possible inflationary spectrum compatible with the obsevational limits[3].

Now we return to the question of which of the background wave functions may admit the deviations of the initial quantum state of gravitons from the vacuum. As we already saw, the Hartle-Hawking wave function ψ_{aHH} are, in a sense , two extremes in the description of the classically forbidden domain. Each of these extrems can be used in eq. (2) as a background wave function. For each of them the solution to eq. (2)can be presented in the form (compare with eq. (5)): $\psi(h,\eta) = C(\eta)e^{-B(\eta)y^2}$. However, the choice of ψ_{HH} or ψ_{aHH} in the classically forbidden region changes the character of solution for the function $B(\eta)$. It is reasonable to restrict the wave function $\psi(h,\eta)$ by the condition of being normalized: $\int_{-infty}^{infty} \psi^* \psi \, dh < \infty$. If this condition is imposed it leads to the restriction $Re\, B(\eta) > 0$. It turns out that this restriction singles out only the vacuum initial state if the background wave function is ψ_{HH} and it leaves the room for non-vacuum initial quantum states if the background wave function is ψ_{aHH}. Thus, if relic gravitational waves are detected with properties different from those following from the

[3]This line is just a low frequency part of the inflationary spectrum presented in Fig. 4).

initial vacuum state one could conclude that the Universe was described by ψ_{aHH}, and not by ψHH, in the classically forbidden regime. This would strenghten the hypothesis that the Universe was created in a quantum process similar to quantum tunneling or decay. Thus, the difference between possible wave functions of the Universe in the classically forbidden regime can be distinguished by exploring the properties of the gravitational wave background existing now.

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FIGURE CAPTIONS

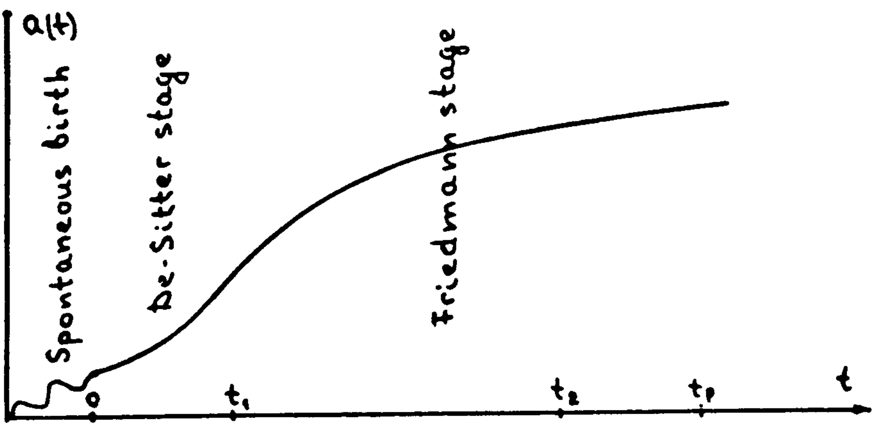

Fig. 1. The scale factor of a complete cosmological theory.

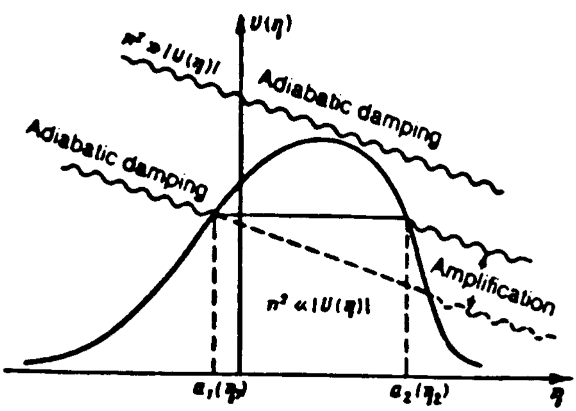

Fig. 2. Parametric (superadiabatic) amplification of waves.

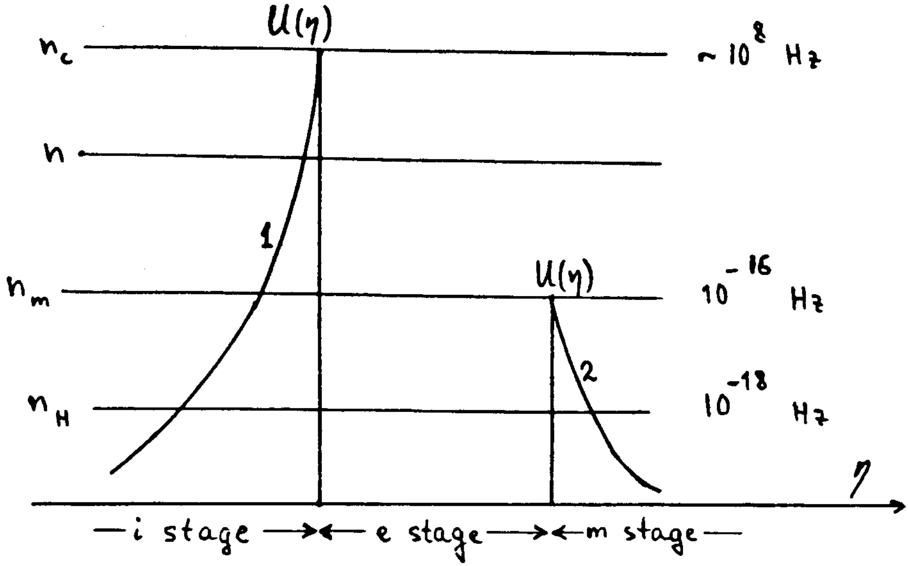

Fig. 3. The potential $U(\eta)$ for the inflationary — radiation-dominated — matter-dominated cosmological model.

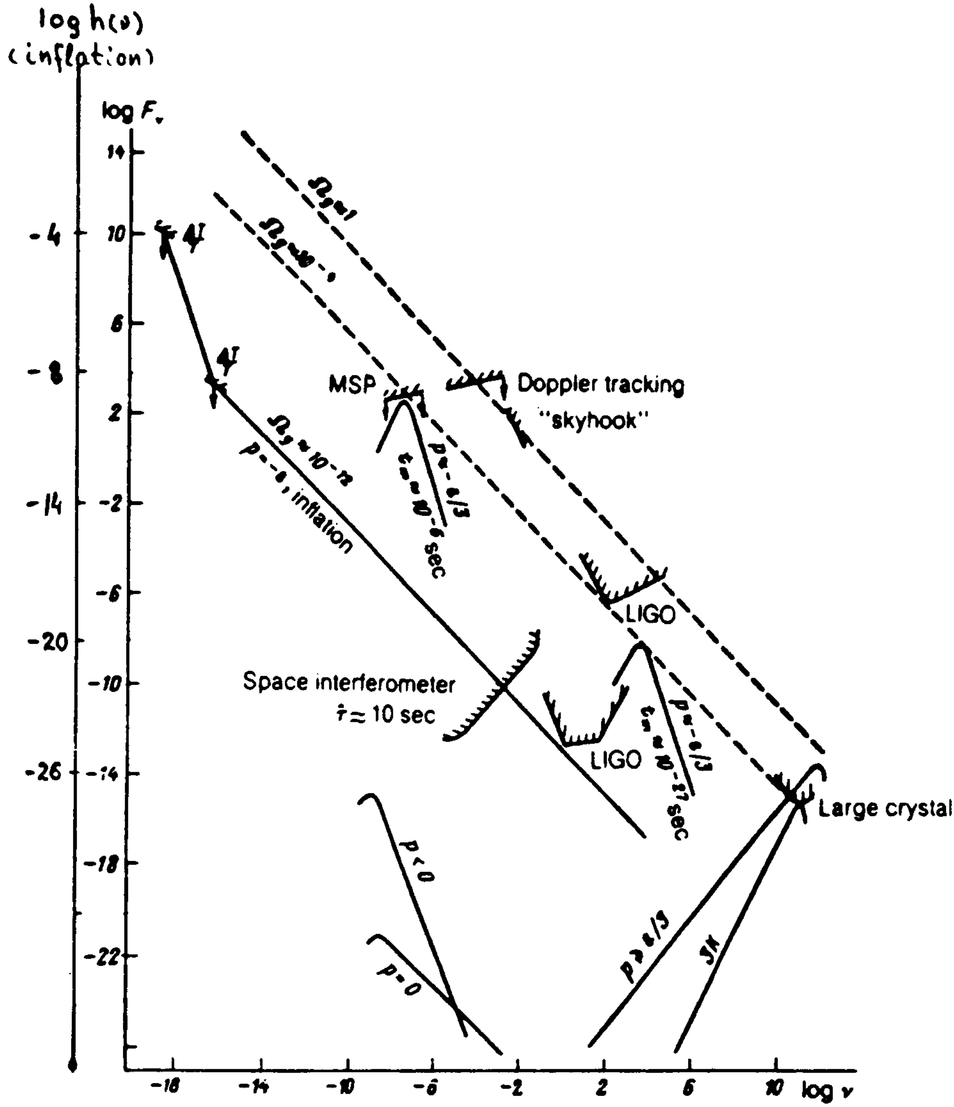

Fig. 4. Theoretical predictions and experimental limits for stochastic gravitational waves.

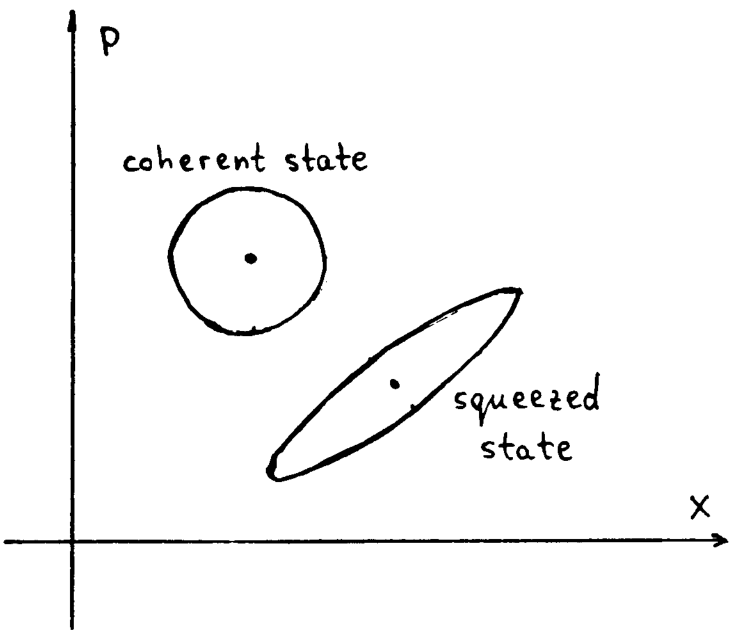

Fig. 5. Variancies for coherent and squeezed quantum states.

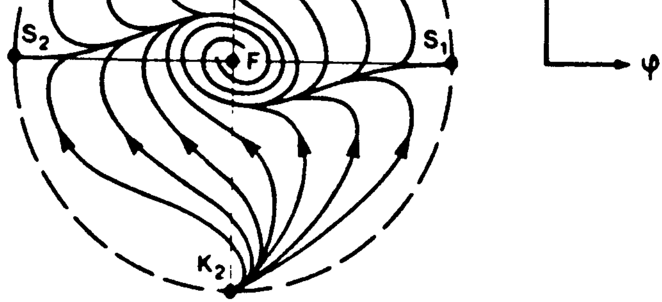

Fig. 6. Classical trajectories at the compactified $(\phi, \dot{\phi})$ phase diagram.

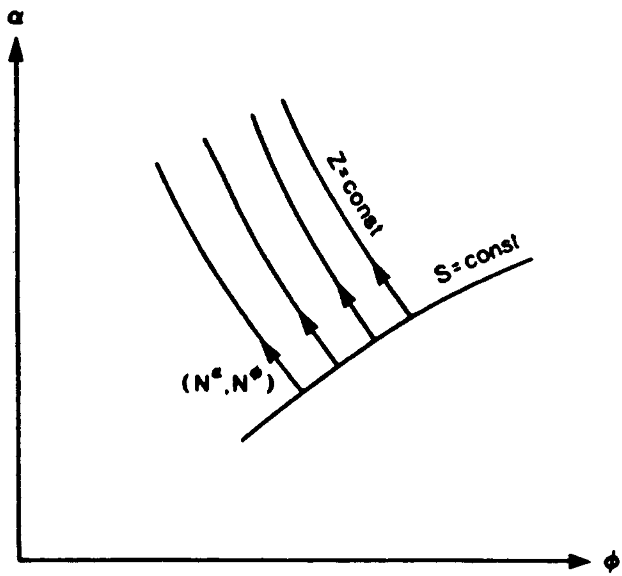

Fig. 7. Classical paths in the (α, ϕ) configuration space.

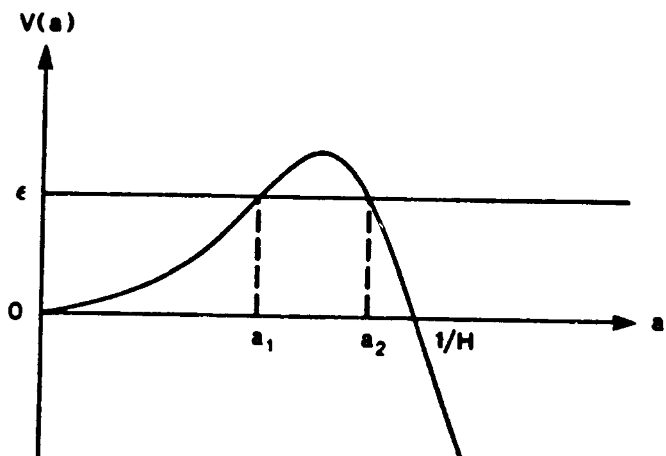

Fig. 8. The potential V(a).

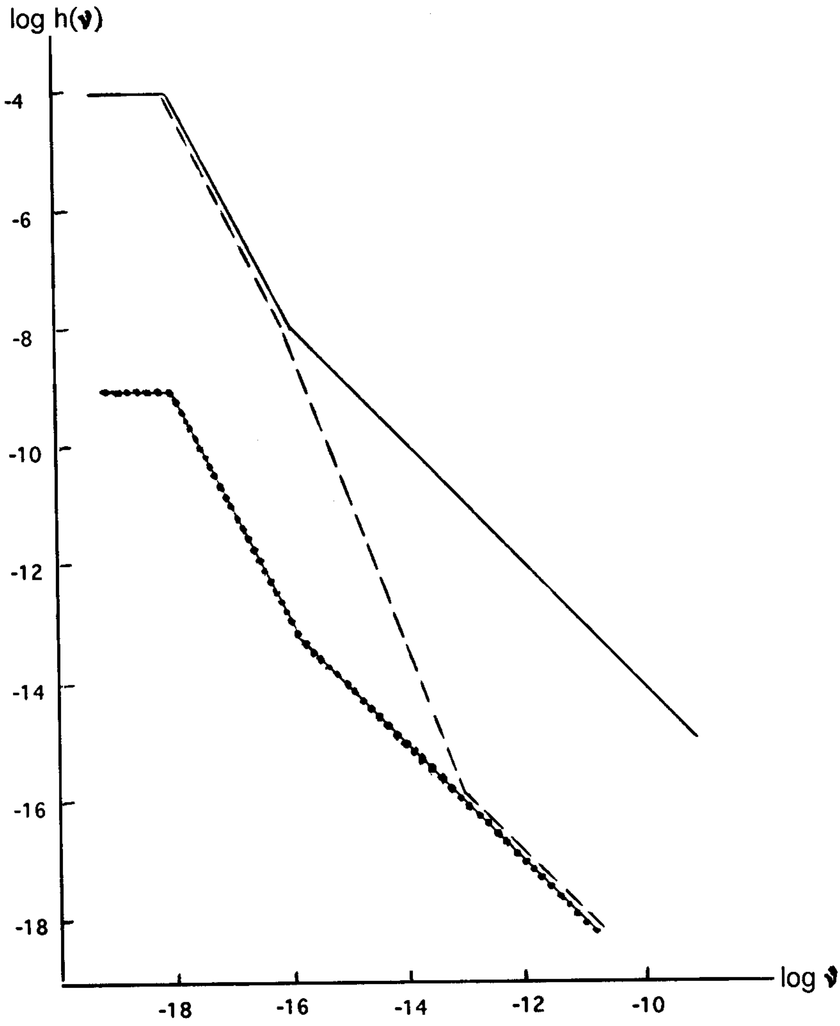

Fig. 9. Possible spectra of relic gravitons.