A Non-Phenomenological Model to Explain Population Growth Behaviors

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Outline

- Proposal: Modeling Population Growth considering microscopic interaction;

- Well-know phenomenological population growth models as special case;

- Comparison with empirical data;
\[
\Delta N(t) = \text{births} - \text{deaths}
\]

**Malthus' Model:**

\[
\frac{dN}{dt} = bN - dN = (b - d)N
\]

\[
N(t) = N_0 e^{(b-d)t}
\]
Crescimento de Bacterias

```
./bacterias.dat
f(x)
```

Número de Bacterias vs. Horas
Verhulst Model

\[
\frac{dN}{dt} = rN(1 - \frac{N}{K}),
\]

\[
N(t) = \frac{N_0Ke^{rt}}{[K + N_0(e^{rt} - 1)]} \rightarrow K \text{ as } t \rightarrow \infty,
\]
The graph shows the growth of yeast over time. The equation given is:

\[ N_t = \frac{13 \cdot 0}{1 + e^{3.32816 - 0.21827t}} \]

This equation represents the amount of yeast, \( N_t \), as a function of time, \( t \). The curve on the graph indicates the growth pattern over 60 hours, with a steady state reached at approximately 13 units of yeast. The term \( K_t \approx 13 \cdot 0 \) at long times suggests asymptotic behavior. The data points show experimental observations, with the line indicating a theoretical model fit.
Examples of Phenomenological Models

- Malthus;
- Verhust;
- Theta-Logistic;
- Gompertz;
- Richard;
- ...
Phenomenological Models
Non-Phenomenological Model: Individual Interactions
Termodynamic and Statistical Physics

macroscopic quantities = pressure, temperature

microscopic quantities = kinetic motion of atoms
  = velocity, momentum
The Model

- Replication rate of the i-th agent:

\[ R = \text{[Self-stimulated replication]} - \text{[competition from field]} \]

The Model

- Replication rate of the i-th agent:

\[ R = \text{[Self-stimulated replication]} - \text{[competition from field]} + \text{[cooperation from field]}. \]
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- Interaction strength decays with distance by: \( \frac{1}{r^\gamma} \)
\[ R_i = k_i - J_1 \sum_{j \neq i} \frac{1}{|r_i - r_j|^{\gamma}} + J_2 \sum_{j \neq i} \frac{1}{|r_i - r_j|^{\gamma}} \]

\[ \frac{d}{dt} N = \sum_{i=1}^{N} R_i \]

\[ \frac{d}{dt} N = N [\langle k \rangle - J_1 I(N) + J_2 I(N)] . \]

\[ I(N) = \frac{\omega_D}{D_f (1 - \frac{\gamma}{D_f})} \left[ \left( \frac{D_f}{\omega_D} N \right)^{1 - \frac{\gamma}{D_f}} - 1 \right] . \]
\[ \frac{d}{dt} N = N \left[ \langle k \rangle - J_1 I(N) + J_2 I(N) \right]. \]

\[ I(N) = \frac{\omega_D}{D_f (1 - \frac{\gamma}{D_f})} \left[ \left( \frac{D_f}{\omega_D} N \right)^{1 - \frac{\gamma}{D_f}} - 1 \right]. \]
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\[ I(N) = \frac{\omega_D}{D_f (1 - \frac{\gamma}{D_f})} \left[ \left( \frac{D_f}{\omega_D} N \right)^{\frac{1-\frac{\gamma}{D_f}}{D_f}} - 1 \right]. \]
The Mukox
The Mukox
\[ \frac{dN}{dt} = rN(1 - N/K), \]

- $\gamma << Df$ : Modelo de Verhulst
When Competition Predominates:

- **Verhulst Model**
- **Richards Model**
- **Gompertz Model**
- **Richards Model**
- **Power Law Growth**
- **Exponential Growth**

$\gamma = 0$

$\gamma = Df$

$\gamma = \gamma^*$

- When **Competition** Predominates:

  - Verhulst Model
  - Richards Model
  - Gompertz Model
  - Power Law Growth

  - $\gamma = 0$
  - $\gamma = Df$
  - $\gamma = \gamma^*$

- When **Cooperation** predominates:

  - Verhulst Model
  - Von Foerster Model
  - Gompertz Model
  - Exponential Growth

  - $\gamma = 0$
  - $\gamma = \gamma^*$
  - $\gamma = Df$
Figure 7: The human population as a function of time since the Middle Ages. The data were obtained from [3] and from the U.S. Census Bureau [34]. The curve is a plot of the equation $N(t) = 65.6(2026 - t)^{-0.78}$ (from Eq. (25)), whose parameters values were obtained via a data fit.
Doomsday: Friday, 13 November, A.D. 2026

At this date human population will approach infinity if it grows as it has grown in the last two millenia.

Heinz von Foerster, Patricia M. Mora, Lawrence W. Amiot

Among the many different aspects which may be of interest in the study of biological populations (1) is the one in which attempts are made to estimate the past and the future of such a population in terms of the number of its elements, if the behavior of this population is observable over a reasonable period of time.

All such attempts make use of two fundamental facts concerning an individual element of a closed biological population—namely, (i) that each element comes into existence by a sexual or asexual process performed by another element of this population ("birth"), and (ii) that after a finite time each element will cease to be a distinguishable member of this population and has to be excluded from the population count ("death").

Under conditions which come close to being paradise—that is, no environmental hazards, unlimited food supply, and no detrimental interaction between elements—the fate of a biological population as a whole is completely determined at all times by reference to the two fundamental properties of an individual element: its fertility and its mortality. Assume, for simplicity, a fictitious population in which all elements behave identically (equivant population, 2) displaying a fertility of \( \gamma_0 \) offspring per element per unit time and having a mortality \( \theta_0 = 1/t_m \), derived from the life span for an individual element of \( t_m \) units of time. Clearly, the elements in the population, is given by

\[
\frac{dN}{dt} = \gamma_0 N - \theta_0 N = a_0 N
\]  

(1)

where \( a_0 = \gamma_0 - \theta_0 \) may be called the productivity of the individual element. Depending upon whether \( a_0 \geq 0 \), integration of Eq. 1 gives the well-known exponential growth or decay of such a population with a time constant of \( 1/a_0 \).

In reality, alas, the situation is not that simple, inasmuch as the two parameters describing fertility and mortality may vary from element to element and, moreover, fertility may have different values, depending on the age of a particular element.

To derive these distribution functions from observations of the behavior of a population as a whole involves the use of statistical machinery of considerable sophistication (3, 4).

However, so long as the elements live in our hypothetical paradise, it is in principle possible, by straightforward mathematical methods, to extract the desired distribution functions, and the fate of the population as a whole, with all its ups and downs, is again determined by properties exclusively attributable to individual elements. If one foregoes the opportunity to describe the behavior of a population in all its temporal details and is satisfied

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Conclusion

- A Microscopic Population Model was proposed;

- Some phenomenological models are reached as special cases;

- Comparassion with empirical data;

- Insights about universal growth behaviours.