Hypergeometric connotations of quantum equations

A. Plastino, M. C. Rocca
National University La Plata

6 de octubre de 2015
The ordinary hypergeometric function $F^2_1(a, b; c; z)$ is a special function represented by the hypergeometric series, that includes many other special functions as specific or limiting cases. It is a solution of a second-order linear ordinary differential equation (ODE). Almost all second-order linear ODEs can be transformed into this equation. Generalized hypergeometric functions include the confluent hypergeometric function as a special case, which in turn have many particular special functions as special instances, such as elementary functions, Bessel functions, and the classical orthogonal polynomials.

The term "hypergeometric series" was first used by John Wallis in his 1655 book Arithmetica Infinitorum. Hypergeometric series were studied by Euler, but the first full systematic treatment was given by Gauss (1813). Studies in the nineteenth century included those of Ernst Kummer (1836), and the fundamental characterisation by Riemann (1857) of the hypergeometric function by means of the differential equation it satisfies. The cases where the solutions are algebraic functions were found by Hermann Schwarz.
\[
\phi(a, b, z) = \sum_{n=0}^{\infty} \frac{a_n}{b_n} \frac{z^n}{n!}; \quad a, b \in \mathbb{R}.
\]

\[a_0 = 1, \quad a_n = a(a + 1)(a + 2) \ldots (a + n - 1); \quad \text{same for } b.\]

A diff. eq. that can be solved using hyperg. functs. is:

\[
z\phi''(a, b; z) + (b - z)\phi'(a, b; z) - a\phi(a, b; z) = 0. \quad (1)
\]
For the special instance $a = b$ we obtain

$$\phi(a, a; z) = e^z,$$  \hspace{1cm} (2)

and (1) now becomes (for $a = b$)

$$z\phi''(a, a; z) + (a - z)\phi'(a, a; z) - a\phi(a, a; z) = 0.$$  \hspace{1cm} (3)

At this point we make a critical choice and set $z$ to

$$z = \frac{i}{\hbar}(px - Et) \text{ and } E = \frac{p^2}{2m},$$  \hspace{1cm} (4)

so that

$$\frac{\partial z}{\partial t} = -\frac{iE}{\hbar}; \quad \frac{\partial z}{\partial x} = \frac{ip}{\hbar}.$$  \hspace{1cm} (5)
Then, on account of (2) one has

\[ e^{\frac{i}{\hbar}(p x - E t)} = \phi \left[ a, a; \frac{i}{\hbar}(p x - E t) \right] \equiv \phi, \quad (6) \]

with \( \frac{\partial \phi}{\partial t} = \frac{E}{i\hbar} \phi \). The associated confluent hypergeometric differential equation is now

\[ \frac{i}{\hbar}(p x - E t)\phi'' + \left[ a - \frac{i}{\hbar}(p x - E t) \right] \phi' - a\phi = 0, \quad (7) \]

\[ \phi' = \frac{d\phi}{dz}; \quad \phi'' = \frac{d^2\phi}{dz^2}. \quad (8) \]
We see from (6) that, since
\[
\frac{\partial \phi}{\partial t} = \phi' \frac{\partial z}{\partial t}; \quad \frac{\partial \phi}{\partial x} = \phi' \frac{\partial z}{\partial x},
\]
then
\[
\phi'' = - \frac{\hbar^2}{p^2} \frac{\partial^2 \phi}{\partial x^2}; \quad \phi' = \frac{i\hbar}{E} \frac{\partial \phi}{\partial t} \equiv \phi,
\]
so that, in the differential equation (7) the two \(a\)–terms cancel each other and the factor \(\frac{i}{\hbar}(px - Et)\) can be simplified. Accordingly, (7) adopts the appearance
\[
- \frac{\hbar^2}{p^2} \frac{\partial^2 \phi}{\partial x^2} - \frac{i\hbar}{E} \frac{\partial \phi}{\partial t} = 0,
\]
or, equivalently (quantum connection),

\[ i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2}. \]  

(12)

Since \( H_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \) we can finally write

\[ i\hbar \frac{\partial \phi}{\partial t} = H_0 \phi, \]  

(13)

i.e., Schrödinger’s free particle equation. For an arbitrary Hamiltonian \( H \), (13) may be generalized to

\[ i \frac{\hbar}{\hbar} \frac{\partial \phi}{\partial t} = H \phi, \]  

(14)

the usual Schrödinger equation (SE). Thus, we deduce Schrödinger’s wave equation (free particle) directly from the hypergeometric differential equation (HDE) and then, by suitable generalization, we infer the usual SE. The “quantumness” is inserted into the HDE via the choice (4).
We start with the critical choice

\[ z = i (kx - \omega t) \quad \omega^2 = k^2 c^2 + \frac{m^2 c^4}{\hbar^2}, \quad (15) \]

and write

\[ e^{i (kx - \omega t)} = \phi [a, a; i(kx - \omega t)] = \phi. \quad (16) \]

The operating confluent hypergeometric differential equation is here

\[ i(kx - \omega t)\phi'' + [a - i(kx - \omega t)] \phi' - a\phi = 0. \quad (17) \]

We now perform similar manipulations of partial derivatives as in the preceding Section and end up with the identities
\[ \phi'' = -\frac{1}{k^2} \frac{\partial^2 \phi}{\partial x^2} \quad \phi' = \frac{i}{\omega} \frac{\partial \phi}{\partial t} = \phi, \quad (18) \]

getting for (17)

\[ -\frac{1}{k^2} \frac{\partial^2 \phi}{\partial x^2} - \frac{i}{\omega} \frac{\partial \phi}{\partial t} = 0. \quad (19) \]

Since

\[ \frac{\partial \phi}{\partial t} = \frac{i}{\omega} \frac{\partial^2 \phi}{\partial t^2}, \quad (20) \]

(19) becomes

\[ -\frac{1}{k^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{i}{\omega^2} \frac{\partial^2 \phi}{\partial t^2} = 0. \quad (21) \]
Using now the equality

\[
\frac{\partial^2 \phi}{\partial x^2} = -k^2 \phi, \tag{22}
\]

and a little algebra we arrive at

\[
\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{m^2 c^2}{\hbar^2} \phi = 0, \tag{23}
\]

the desired Klein-Gordon equation.
A new nonlinear Schrödinger equation has been recently advanced by Nobre, Rego, Monteiro and Tsallis. The NRT proposal constitutes an intriguing contribution to a line of enquiry that has been the focus of continuous research activity for several years: the exploration of nonlinear versions of some of the fundamental equations of physics. In particular, nonlinear versions of the Schrödinger equation have found important applications in various areas. The most studied nonlinear Schrödinger equation involves a cubic nonlinearity in the wave function. In classical contexts this nonlinear equation has been applied to fibre optics and also to the study of water waves. In quantum mechanical scenarios this kind of equations usually govern the behaviour of a single-particle wave function that provides an effective, mean-field description of a quantum many-body system. A celebrated example of this approach is given by the Gross Pitaevskii equation, used in the study of Bose Einstein condensates.
Pochhammer symbol
≡ \( x(n) = x(x + 1)(x + 2) \cdots (x + n - 1). \)

\[
F^2_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{a(n)b(n)}{c(n)} \frac{z^n}{n!}; \quad (|z| < 1), \tag{24}
\]

where the series terminates if either \( a \) or \( b \) is a non-zero integer. Special case:

\[
F(-m, b, b, -z) = (1 + z)^m. \tag{25}
\]
q-Exponential function

\[ e_q(x) = [1 + (q - 1)x]^{\frac{1}{1-q}}, \quad (26) \]

with \( 1 + (q - 1)x > 0 \). It is \( e_q(x) = 0 \) otherwise. Also, \( q \in \mathcal{R} \).

In particular, for \( e_q[(i/\hbar)(px - Et)] \), this relation holds (with \( E = \frac{p^2}{2m} \))

\[ e_q[(i/\hbar)(px - Et)] = F \left[ \frac{1}{q-1}, \gamma; \gamma; \frac{i}{\hbar}(q - 1)(px - Et) \right] \quad (27) \]

which is a fundamental result for us.
The hypergeometric function obeys the following, nonlinear, differential equation

\[ z(1-z)F'' + [\gamma - (\alpha + \beta + 1)z]F' - \alpha\beta F = 0. \]  (28)

Choose \( z = px - Et \) and abbreviate

\[ F \equiv F \left[ \frac{1}{q-1}, \gamma; \gamma; \frac{i}{\hbar} (q-1)(px - Et) \right]. \]  (29)

\[ -\frac{\hbar^2}{2m} F^{(1-q)} \frac{\partial^2}{\partial x^2} F - i\hbar q \frac{\partial}{\partial t} F = 0, \]  (30)

that can be rewritten as

\[ i\hbar q \frac{\partial}{\partial t} F = F^{(1-q)} H_0 F, \]  (31)

\( H_0 = (-\hbar^2/2m)(\partial^2/\partial x^2). \) Note that, for \( q = 1 \), one reobtains Schrödinger’s free particle equation.
Remember

\[ e_q[(i/\hbar)(px - Et)] = F \left[ \frac{1}{q-1}, \gamma; \gamma; \frac{i}{\hbar}(q - 1)(px - Et) \right] \]

Set our Hyperg. Fnct. \( F = \psi(x, t) \equiv F \equiv e_q[(i/\hbar)(px - Et)]. \) \hspace{1cm} (32)

Eq. (31) becomes

\[ i\hbar q \psi(x, t)(q-1) \frac{\partial}{\partial t} \psi(x, t) = H_0 \psi(x, t); \]

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t)^q = H_0 \psi(x, t). \] \hspace{1cm} (33)
\[ \phi(x, t) = \psi(x, t)^q, \quad (34) \]

\[ i\hbar \frac{\partial \phi(x, t)}{\partial t} = H_0 \phi(x, t)^{1/q}. \quad (35) \]

Generalize to
\[ i\hbar \frac{\partial \phi(x, t)}{\partial t} = H\phi(x, t)^{1/q}; \quad i\hbar \frac{\partial \psi(x, t)^q}{\partial t} = H\psi(x, t). \quad (36) \]

NRT Eq.
\[ i\hbar (2 - q) \frac{\partial}{\partial t} \psi(x, t) = H[\psi(x, t)]^{2-q}. \quad (37) \]
Consider now a time-independent $H$. A separable situation ensues.

$$\psi(x, t) = f(t)g(x). \quad (38)$$

Then, (36) becomes

$$i\hbar [g(x)]^q \frac{d}{dt} [f(t)]^q = f(t)Hg(x). \quad (39)$$

$$i\hbar f(t)^{-1} \frac{d}{dt} f(t)^q = g(x)^{-q}Hg(x) = \lambda = \text{Const.}, \quad (40)$$

$$i\hbar \frac{d}{dt} f(t)^q = \lambda f(t), \quad (41)$$

$$Hg(x) = \lambda g(x)^q. \quad (42)$$
Free particle - Separation of variables

\[ i\hbar \frac{d}{dt} [f(t)]^q = \lambda f(t) \]  \hspace{1cm} (43)

\[- \frac{\hbar^2}{2m} \frac{d}{dx^2} g(x) = \lambda [g(x)]^q \]  \hspace{1cm} (44)

A possible solution is

\[ E = \frac{p^2}{2m} = \lambda \]  \hspace{1cm} (45)

\[ f(t) = \left[ 1 + \frac{i}{\hbar} \frac{(1 - q)}{q} \frac{1}{E \hbar t} \right]^{\frac{1}{q-1}} \]  \hspace{1cm} (46)

\[ g(x) = \left[ 1 + \frac{i}{\hbar} \frac{(1 - q)}{\sqrt{2(q + 1)}} \frac{1}{p \hbar x} \right]^{\frac{2}{1-q}} . \]  \hspace{1cm} (47)
Non-linear Klein-Gordon Equation

\[ e_q(z) = [1 + i(1 - q)(kx - \omega t)]^{\frac{1}{1-q}}. \] (48)

\[ F[\frac{1}{q-1}, \gamma; \gamma; i(q - 1)(kx - \omega t)] \equiv F. \] (49)

We have the equalities \((z = i(kx - \omega t))\)

\[ F'' = -\frac{1}{k^2(q - 1)^2} \frac{\partial^2 F}{\partial x^2}, \] (50)

\[ F' = -\frac{1}{i\omega(q - 1)} \frac{\partial F}{\partial t} = -\frac{F^{(1-q)}}{\omega^2 q(q - 1)} \frac{\partial^2 F}{\partial t^2}, \] (51)

\[ F = \frac{i}{\omega} F^{(1-q)} \frac{\partial F}{\partial t}, \] (52)
Some heavy algebra leads to

\[
\frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} + \frac{qm^2 c^2}{\hbar^2} F^{(2q-1)} = 0.
\]  

(53)

If \( \phi \) is given by

\[
\phi(x, t) = A \left[ 1 + i(1 - q)(kx - \omega t) \right]^{\frac{1}{1-q}},
\]  

(54)

we find, via (53),

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} [\phi(x, t)] - \frac{\partial^2}{\partial x^2} [\phi(x, t)] + \frac{qm^2 c^2}{\hbar^2} [\phi(x, t)]^{(2q-1)} = 0,
\]  

(55)
We want to tackle

\[ i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2}, \]

(56)

via the Gaussian packet

\[ \psi(x, t) = e^{-[a(t)x^2+b(t)x+c(t)]}, \]

(57)

with the initial condition \( \psi(0, 0) = 1 \), entailing \( c(0) = 0 \).
The q-Gaussian wave packet

\[ i\hbar \frac{\partial \psi^q}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \]  

(58)

We propose as a solution

\[ \psi = \left\{ 1 + (q - 1) \left[ a(t)x^2 + b(t)x + c(t) \right] \right\}^{\frac{1}{1-q}}, \]  

(59)

where \( a, b, \) and \( c \) are temporal functions to be determined.
The solution is given by

\[ a(t) = \frac{mq}{i\hbar(q + 1)t + mq\alpha}, \quad (60) \]

\[ b(t) = \frac{1}{\beta} \frac{1}{[i\hbar(q + 1)t + mq\alpha]}, \quad (61) \]

\[ c(t) = (mq\alpha)^{\frac{1-q}{1+q}} \left( \frac{1}{q - 1} - \frac{1}{4m^2q^2\beta^2\alpha} \right) \frac{q-1}{q+1} + \frac{1}{4mq\beta^2[i\hbar(q + 1)t + mq\alpha]} + \frac{1}{1 - q}, \quad (62) \]

where \( \alpha \) and \( \beta \) are constants to be fixed according to initial or boundary conditions for (58).