deformed binomial distributions

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foundations of complexity - rio de janeiro - 2015
• uncorrelated system - binomial distribution
• correlated system - deformed binomial distribution
• mathematical formalism to construct a deformed binomial distribution
• extensivity - analytical studies of Boltzmann-Gibbs, Tsallis and Rényi entropies for some deformed binomial distributions
• extensivity and correlation - discussion
“I prefer going to the ancient languages for the names of important scientific quantities, so that they mean the same thing in all living tongues. I propose, accordingly, to call S the entropy of a body, after the Greek word \(\tau rho\pi\eta\), ‘transformation’. I have designedly coined the word entropy to be similar to energy, for these two quantities are so analogous in their physical significance, that an analogy of denominations seems to me helpful”

R. Clausius, Annalen der Physik und Chimie 7 (1865) 23
1) Die Energie der Welt ist konstant
2) Die Entropie der Welt strebt einem Maximum zu

\[ dS = \frac{dQ}{T} \quad (59) \]

\[ S = S_0 + \int \frac{dQ}{T} \quad (60) \]

- macroscopic quantity
Über die Beziehung zwischen dem zweiten Hauptsätze des mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung, respective den Sätzen über das Wärmegleichgewicht.

On the relationship between the second main theorem of mechanical heat theory and the probability calculation with respect to the results about the heat equilibrium.

Von dem c. M. Ludwig Boltzmann in Graz
• μ-space
• partitioned into many small and disjoint (6-dimensional) rectangular cells, each one having energy 0, ε, 2ε, 3ε, ⋅⋅⋅, pε
• filled with points representing the N particles that comprise the gas, where each possible distribution is called a complexion \( \{ w_i \} \)
• macrostate of the gas can then be described as the number of particles that occupy each of these rectangular regions of the μ-space

\[
\sum_{i=0}^{P} w_i = N \quad \sum_{i=0}^{P} w_i i \varepsilon = \lambda \varepsilon
\]

\[
P \propto \frac{N!}{w_0! w_1! \cdots w_p!}
\]

\[
W = \frac{P}{\sum P} \log P \sim - \sum_i w_i \log w_i
\]

\[
w_j \propto \exp \left( -j \varepsilon / \bar{\varepsilon} \right) \quad \bar{\varepsilon} = \frac{\lambda \varepsilon}{N}
\]

Boltzmann related log P with the expression of 1872’s paper and with Clausius entropy
• Boltzmann suggests at the end of the paper that the same argument might be applicable also to dense gases and even to solids

• the assumption that the total energy can be expressed in the form $E = \Sigma_i n_i \varepsilon_i$ means that the energy of each particle depends only on the cell in which it is located, and not the state of other particles. This can only be maintained, independently of the number $N$, if there is no interaction at all between the particles. The validity of the argument is thus really restricted to ideal gases

• beginning of statistical mechanics
A Mathematical Theory of Communication

By C. E. SHANNON

INTRODUCTION

The recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist\(^1\) and Hartley\(^2\) on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and the savings possible due to the statistical structure of the original message and due to the nature of the final destination of the information.
6. **Choice, Uncertainty and Entropy**

We have represented a discrete information source as a Markoff process. Can we define a quantity which will measure, in some sense, how much information is "produced" by such a process, or better, at what rate information is produced?

Suppose we have a set of possible events whose probabilities of occurrence are $p_1, p_2, \cdots, p_n$. These probabilities are known but that is all we know concerning which event will occur. Can we find a measure of how much "choice" is involved in the selection of the event or of how uncertain we are of the outcome?

If there is such a measure, say $H(p_1, p_2, \cdots, p_n)$, it is reasonable to require of it the following properties:

1. $H$ should be continuous in the $p_i$.

2. If all the $p_i$ are equal, $p_i = \frac{1}{n}$, then $H$ should be a monotonic increasing
function of \( n \). With equally likely events there is more choice, or uncertainty, when there are more possible events.

3. If a choice be broken down into two successive choices, the original \( H \) should be the weighted sum of the individual values of \( H \). The meaning of this is illustrated in Fig. 6. At the left we have three possibilities \( p_1 = \frac{1}{2}, p_2 = \frac{1}{3}, p_3 = \frac{1}{6} \). On the right we first choose between two possibilities each with probability \( \frac{1}{2} \), and if the second occurs make another choice with probabilities \( \frac{2}{3}, \frac{1}{3} \). The final results have the same probabilities as before. We require, in this special case, that

\[
H(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}) = H(\frac{1}{2}, \frac{1}{2}) + \frac{1}{2}H(\frac{2}{3}, \frac{1}{3})
\]

The coefficient \( \frac{1}{2} \) is because this second choice only occurs half the time.

![Fig. 6—Decomposition of a choice from three possibilities.](image)

In Appendix II, the following result is established:

**Theorem 2**: The only \( H \) satisfying the three above assumptions is of the form:

\[
H = -K \sum_{i=1}^{n} p_i \log p_i
\]

where \( K \) is a positive constant.
Information Theory and Statistical Mechanics

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(Received September 4, 1956; revised manuscript received March 4, 1957)

Information theory provides a constructive criterion for setting up probability distributions on the basis of partial knowledge, and leads to a type of statistical inference which is called the maximum-entropy estimate. It is the least biased estimate possible on the given information; i.e., it is maximally noncommittal with regard to missing information. If one considers statistical mechanics as a form of statistical inference rather than as a physical theory, it is found that the usual computational rules, starting with the determination of the partition function, are an immediate consequence of the maximum-entropy principle. In the resulting “subjective statistical mechanics,” the usual rules are thus justified independently of any physical argument, and in particular independently of experimental verification; whether or not the results agree with experiment, they still represent the best estimates that could have been made on the basis of the information available.

It is concluded that statistical mechanics need not be regarded as a physical theory dependent for its validity on the truth of additional assumptions not contained in the laws of mechanics (such as ergodicity, metric transitivity, equal a priori probabilities, etc.). Furthermore, it is possible to maintain a sharp distinction between its physical and statistical aspects. The former consists only of the correct enumeration of the states of a system and their properties; the latter is a straightforward example of statistical inference.
would give us a reason for preferring one probability distribution over another in cases where both agree equally well with the available information.

For further discussion of this problem, one must recognize the fact that probability theory has developed in two very different directions as regards fundamental notions. The “objective” school of thought\textsuperscript{8,9} regards the probability of an event as an objective property of that event, always capable in principle of empirical measurement by observation of frequency ratios in a random experiment. In calculating a probability distribution the objectivist believes that he is making

\textsuperscript{8} Yet this is precisely the problem confronting us in statistical mechanics; on the basis of information which is grossly inadequate to determine any assignment of probabilities to individual quantum states, we are asked to estimate the pressure, specific heat, intensity of magnetization, chemical potentials, etc., of a macroscopic system. Furthermore, statistical mechanics is amazingly successful in providing accurate estimates of these quantities. Evidently there must be other reasons for this success, that go beyond a mere correct statistical treatment of the problem as stated above.

\textsuperscript{9} The problems associated with the continuous case are fundamentally more complicated than those encountered with discrete random variables; only the discrete case will be considered here.

\textsuperscript{7} For several examples, see E. P. Northrop, \textit{Riddles in Mathematics} (D. Van Nostrand Company, Inc., New York, 1944), Chap. 8.


the subjective point of view.

Just as in applied statistics the crux of a problem is often the devising of some method of sampling that avoids bias, our problem is that of finding a probability assignment which avoids bias, while agreeing with whatever information is given. The great advance provided by information theory lies in the discovery that there is a unique, unambiguous criterion for the “amount of uncertainty” represented by a discrete probability distribution, which agrees with our intuitive notions that a broad distribution represents more uncertainty than does a sharply peaked one, and satisfies all other conditions which make it reasonable.\textsuperscript{4}

In Appendix A we sketch Shannon’s proof that the quantity which is positive, which increases with increasing uncertainty, and is additive for independent sources of uncertainty, is

\[ H(p_1 \cdots p_n) = -K \sum_i p_i \ln p_i, \]  

(2-3)

where \( K \) is a positive constant. Since this is just the expression for entropy as found in statistical mechanics, it will be called the entropy of the probability distribution \( p_i \); henceforth we will consider the terms “entropy” and “uncertainty” as synonymous.

\textsuperscript{10} J. M. Keynes, \textit{A Treatise on Probability} (MacMillan Company, London, 1921).

Rényi entropy

ON MEASURES OF ENTROPY AND INFORMATION

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1. Characterization of Shannon’s measure of entropy

Let $\varphi = (p_1, p_2, \ldots, p_n)$ be a finite discrete probability distribution, that is, suppose $p_k \geq 0 (k = 1, 2, \ldots, n)$ and $\sum_{k=1}^{n} p_k = 1$. The amount of uncertainty (1.20) because (1.20) is much weaker. As a matter of fact there are many quantities other than (1.1) which satisfy the postulates (a), (b), (c), and (1.20). For instance, all the quantities

\begin{equation}
H_\alpha(p_1, p_2, \ldots, p_n) = \frac{1}{1 - \alpha} \log_2 \left( \sum_{k=1}^{n} p_k^\alpha \right),
\end{equation}

where $\alpha > 0$ and $\alpha \neq 1$ have these properties. The quantity $H_\alpha(p_1, p_2, \ldots, p_n)$ (Feinstein [3]). Fadeev’s postulates are as follows.

http://projecteuclid.org/euclid.bsmsp/1200512181
ENTROPY, PROBABILITY AND DYNAMICS

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Abstract

Boltzmann’s struggle with a derivation of the Second Law of Thermodynamics is sketched. So is his first derivation of the connection between entropy and probability in 1877. Planck’s derivation and quantum mechanical modifications of Boltzmann’s connection between en...

case. This allows $S$ to be determined free of an arbitrary constant and at the same time to connect the classical and quantum mechanical values of $S$, such that $S = 0$ at $T = 0$ (Sackur-Tetrode formula)\[^9\].

To the best of my knowledge neither Boltzmann’s nor Planck’s derivation of $S$ for an ideal gas has ever been generalized to an interacting gas. Results for such a gas can be obtained from Gibbs’ microcanonical ensemble but that is based ultimately on the unproven ergodic hypothesis.
one macroscopic entropy ...

many entropic forms depending on the probability of the microstates

BG, Rényi, Tsallis, Kaniadakis, Abe, Hanel-Thurner, ...
independent systems
uncorrelated system - binomial distribution

- \( n \) independent trials with two possible outcomes - "win" or "loss"

\[ \eta \in [0, 1] \]

\[
p_k^n(\eta) = \binom{n}{k} \eta^k (1 - \eta)^{n-k} = \frac{n!}{(n-k)! \, k!} \eta^k (1 - \eta)^{n-k}
\]

probability to have \( k \) wins in \( n \) trials - regardless the order

\[
\varpi_k^n = \eta^k (1 - \eta)^{n-k} \quad \Rightarrow \quad \varpi_{k-1}^{n-1} = \varpi_k^n + \varpi_{k+1}^n
\]

Leibniz triangle rule
Pascal and Leibniz rules
limit of large $n$

\[
\binom{n}{k} \xrightarrow{k=n\eta} \binom{n}{n\eta} \sim \frac{1}{\sqrt{2\pi n\eta(1 - \eta)}} \exp\left[ n \left( -x \log\frac{x}{\eta} - (1 - x) \log\frac{1 - x}{1 - \eta} \right) \right]
\]

\[
\eta^{n\eta}(1 - \eta)^{n(1 - \eta)} \rightarrow \exp\left[ n \left( x \log \eta + (1 - x) \log [1 - \eta] \right) \right]
\]

\[
\binom{n}{k} \eta^k (1 - \eta)^{n-k} \sim \frac{1}{\sqrt{2\pi n\eta(1 - \eta)}} \exp\left[ n \left( -x \log\frac{x}{\eta} - (1 - x) \log\frac{1 - x}{1 - \eta} \right) \right]
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \eta^k (1 - \eta)^{n-k} \xrightarrow{k=n\eta} n \int_{0}^{1} dx \frac{1}{\sqrt{2\pi n\eta(1 - \eta)}} \exp[nf(x, \eta)]
\]

Laplace's method

\[
\xrightarrow{\text{Laplace's method}} n \int_{-\infty}^{\infty} dx \exp \left[ -\frac{n}{2\eta(1 - \eta)} (x - \eta)^2 \right] = 1
\]

\[
\Rightarrow \mathfrak{P}_k^{(n)} = \binom{n}{k} \eta^k (1 - \eta)^{n-k} \rightarrow \sqrt{\frac{n}{2\pi \eta(1 - \eta)}} \exp \left[ -\frac{n}{2\eta(1 - \eta)} (x - \eta)^2 \right]
\]

limit distribution is a Gaussian
binomial case

\[ P_k^{(n)} = \binom{n}{k} \eta^k (1 - \eta)^{n-k} \quad \eta \in [0, 1] \]

\[ \sum_{k=0}^{n} P_k^{(n)} = 1 \]

\[ \omega_k^{(n)} = \frac{P_k^{(n)}}{\binom{n}{k}} = \eta^k (1 - \eta)^{n-k} \]

\[ \sum_{k=0}^{n} \binom{n}{k} k \omega_k^{(n)} (\eta) = n \eta = <k> \]

\[ S_{BG}(\eta) = - \sum_{k=0}^{n} \binom{n}{k} \omega_k^{(n)} \log(\omega_k^{(n)}) = - \sum_{k=0}^{n} P_k^{(n)} \log(\omega_k^{(n)}) \]

\[ S_{BG}(\eta) = -(<k> \log \eta + <n - k> \log(1 - \eta)) \]

\[ = -n [\eta \log \eta + (1 - \eta) \log(1 - \eta)] \leq S_{BG}(1/2) = n \log 2 \]

\[ S_{BG} \text{ is extensive} \]
binomial case - $S_q$ entropy

$$S_q = \frac{1 - \sum_i p_i^q}{q - 1} \rightarrow S_q = \frac{1 - \sum_k \binom{n}{k} \left( \omega_k^{(n)} \right)^q}{q - 1}$$

$$\sum_{k=0}^{n} \binom{n}{k} \left( \omega_k^{(n)} \eta \right)^q = \sum_{k=0}^{n} \binom{n}{k} \eta^{qk} (1 - \eta)^{q(n-k)} = (\eta^q + (1 - \eta)^q)^n$$

$$= \exp \left[ n \log (\eta^q + (1 - \eta)^q) \right]$$

0 < $q$ < 1 \rightarrow $S_q \propto \exp \left[ n \log (\eta^q + (1 - \eta)^q) \right]$

$q$ > 1 \rightarrow $S_q \rightarrow \frac{1}{q - 1}$

$S_q$ is not extensive
systems with independent events —

binomial

\[ \binom{n}{k} q^k (1 - q)^{n-k} \rightarrow S_{BG} = -\sum_{k=0}^{n} p_{k}^{(n)} \log \frac{p_{k}^{(n)}}{\binom{n}{k}} \sim n \log 2 \]

\[ p_{n,k} \equiv \frac{p_{k}^{(n)}}{\binom{n}{k}} \]

\[ S_{q} = \frac{1 - \sum_{k=0}^{n} \binom{n}{k} p_{n,k}^{q}}{q - 1} \]

\[ N(t) = e^{t} \]

\[ \eta = 1/2 \]

 BG linear with n!
binomial case – Rényi entropy

\[ S_R^{(q)}(\eta) = \frac{1}{1 - q} \log \left[ \sum_{k=0}^{n} \binom{n}{k} (w_k^{(n)})^q \right] \]

\[ \sum_{k=0}^{n} \binom{n}{k} (w_k^{(n)}(\eta))^q = \sum_{k=0}^{n} \binom{n}{k} \eta^{qk} (1 - \eta)^{q(n-k)} = (\eta^q + (1 - \eta)^q)^n \]

\[ = \exp [n \log (\eta^q + (1 - \eta)^q)] \]

\[ S_R^{(q)}(\eta) = \frac{n}{1 - q} \log [\eta^q + (1 - \eta)^q] \leq S_R^{(q)}(1/2) = n \log 2 \]

\[ (0 < q < 1) \]

\( S_R \) is extensive as well!
deformed binomial distribution => correlation between events
Laplace – de Finetti modification of the binomial law

\[ \mathcal{P}_k^{(n)} = \binom{n}{k} \mathcal{W}_k^{(n)} \]

binomial \( \rightarrow \mathcal{W}_k^{(n)} = \eta^k (1 - \eta)^{n-k} \)

\[ \tilde{\mathcal{P}}_k^{(n)} := \binom{n}{k} \tilde{\mathcal{W}}_k^{(n)} \]

binary correlated system

\[ \tilde{\mathcal{W}}_k^{(n)} := \int_0^1 dy \, y^k (1 - y)^{n-k} g(y) \quad \text{where} \quad \int_0^1 dy \, g(y) = 1 \]

\[ \tilde{\mathcal{W}}_k^{(n-1)} = \tilde{\mathcal{W}}_k^{(n)} + \tilde{\mathcal{W}}_{k+1}^{(n)} \]

Leibniz triangle rule

binary exchangeable stochastic process
limit distribution and extensivity

\( \rho(x) = g(x) \)

Boltzmann-Gibbs entropy is extensive

Hanel, Thurner, Tsallis, EPJB (2009)
Rényi

\[ \tilde{S}_R^{(q)} [g] = \frac{1}{1 - q} \log \left[ \sum_{k=0}^{n} \binom{n}{k} \left( \tilde{\omega}_k^{(n)} \right)^q \right] \]

\[ \tilde{S}_R^{(q)} [g] \sim n \log 2 \]

\[ g(y) = \frac{8}{\pi} \sqrt{y(1-y)} \]

\[ q = 1/2 \]
two microscopic entropies are extensive for the Laplace-de Finetti case
systems "more correlated"
several kind of deformations have been studied:

- G. Sicuro
correlated events - deformed binomial

- $n$ correlated trials with two possible outcomes - "win" or "loss"

$x_0, x_1, x_2, x_3, \cdots, x_n, \cdots$  \quad $x_0 = 0$  \quad $x_n > x_{n-1}$

$x_n! = x_1 x_2 \cdots x_n$,  \quad $x_0! \equiv 1$

\[
\begin{align*}
\mathbb{P}_k^{(n)}(\eta) &= \frac{x_n!}{x_{n-k}! x_k!} q_k(\eta) q_{n-k} (1 - \eta) \\
&= \frac{x_n!}{x_{n-k}! x_k!} \eta^k (1 - \eta)^{n-k}
\end{align*}
\]

deformed probability to have $k$ wins in $n$ trials - induced by correlations

\[
\forall n \in \mathbb{N}, \quad \sum_{k=0}^{n} \mathbb{P}_k^{(n)}(\eta) = 1. \quad (x_n \to n \Rightarrow q_k \to \eta^k)
\]
probabilistic interpretation

$q_k(\eta)$ has to be nonnegative for all $\eta \in [0, 1]$
program

- choose a sequence: \( x_0, x_1, x_2, \ldots, x_n, \ldots \)
- construct the generalized exponential \( N(t) \)
  \[
  N(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!}
  \]
- construct the functions \( q_n(\eta) \) using
  \[
  q_n(\eta) + q_n(1-\eta) = \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k \end{array} \right) q_k(\eta) q_{n-k}(1-\eta)
  \]
  \[
  q_0(\eta) = 1
  \]
  \[
  q_1(\eta) = \eta
  \]
- construct the function \( p_k^{(n)}(\eta) \)
- this procedure does not work in general!
example of the wrong route

\[ x_0 = 0, \ x_1 = 1 - \epsilon, \ x_2 = 2 - \epsilon, \ldots, \ x_n = n - \epsilon, \ldots \]
solving the positiveness problem by means of generating functions

\[ p_k^{(n)}(\eta) = \frac{x_n!}{x_k! x_{n-k}!} q_k(\eta) q_{n-k}(1 - \eta) \]

\[ \eta \rightarrow 1 - \eta \quad k \rightarrow n - k \quad \Rightarrow \quad \text{symmetric distributions} \]

\[ \sum_{k=0}^{n} p_k^{(n)}(\eta) = 1 \quad \left( N(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!} \right) \]
\[ \forall n, k \in \mathbb{N}, \quad \forall \eta \in [0, 1], \quad p^{(n)}_k(\eta) \geq 0 \]

\[ G(\eta; t) := \sum_{n=0}^{\infty} \frac{q_n(\eta)}{x_n!} t^n \]

\[ N(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!} \]

\[ \sum_{k=0}^{n} p^{(n)}_k(\eta) = 1 \]

\[ p^{(n)}_k(\eta) = \frac{x_n!}{x_{n-k}! x_k!} q_k(\eta) q_{n-k} (1 - \eta) \]

\[ G(\eta; t) G(1 - \eta; t) = N(t) \]

\[ N(t) = \left( \sum_{k=0}^{\infty} \frac{q_k(\eta) t^k}{x_k!} \right) \left( \sum_{m=0}^{\infty} \frac{q_m(1 - \eta) t^m}{x_m!} \right) \]

\[ = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_k(\eta) q_m(1 - \eta)}{x_k! x_m!} t^{k+m} \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{x_n!}{x_k! x_{n-k}!} q_k(\eta) q_{n-k} (1 - \eta) \right) \frac{t^n}{x_n!} \]
\[ \forall n, k \in \mathbb{N}, \quad \forall \eta \in [0, 1], \quad p_k^{(n)}(\eta) \geq 0 \]

\[ G(\eta; t) := \sum_{n=0}^{\infty} \frac{q_n(\eta)}{x_n!} t^n \quad G(\eta; t) G(1 - \eta; t) = \mathcal{N}(t) \]

\[ G(\eta; t) = \pm \sqrt{\mathcal{N}(t)} e^{\Phi(\eta, 1-\eta; t)} \quad \left( \mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!} \right) \]

\[ \Phi(x, y; t) = -\Phi(y, x; t) \]

**simplest case:** \[ \Phi(x, y; t) = (x - y) \varphi(t) \]

\[ G(0; t) = 1 \iff G(1; t) = \mathcal{N}(t) \]

\[ \mathcal{N}(t) = e^{2\Phi(1, 0; t)} = e^{2\varphi(t)} \quad G(\eta; t) = e^{(2\eta)\varphi(t)} = (\mathcal{N}(t))^\eta \]

\[ \forall \eta \in [0, 1], \quad \mathcal{N}(t)^\eta = \sum_{n=0}^{\infty} q_n(\eta) \frac{t^n}{x_n!} \]
solving the positiveness problem by means of generating functions

\[ p_k^{(n)}(\eta) = \frac{x_n!}{x_k! x_{n-k}!} q_k(\eta) q_{n-k} (1 - \eta) \]

\[ \eta \rightarrow 1 - \eta \quad k \rightarrow n - k \quad \Rightarrow \text{symmetric distributions} \]

\[ \sum_{k=0}^{n} p_k^{(n)}(\eta) = 1 \quad \sum_{n=0}^{\infty} \frac{q_n(\eta)}{x_n!} t^n \]

\[ \Sigma \]

all \( a_n > 0, \ n \geq 2 \)

\[ \Sigma_+ = \{ \mathcal{N} \in \Sigma \mid \forall \eta \in [0, 1[, \ q_n(\eta) > 0 \} = \{ e^F \mid F \in \Sigma_0 \} \]

main theorem:

\[ \Sigma_0 \]

\[ (F(t) = a_1 t + a_2 t^2 + \cdots) \]

\[ > 0 \quad \geq 0 \]

JMP 53 (2012)

JMP 54 (2013)
example 1 - q-exponential

\[ \mathcal{N}(t) = \left(1 - \frac{t}{\alpha}\right)^{-\alpha}, \quad \alpha > 0 \]

\( x_n! = \alpha^n \frac{\Gamma(\alpha)n!}{\Gamma(n + \alpha)} = \frac{\alpha^n n!}{(\alpha)_n} \quad \Rightarrow \quad x_n = \frac{n\alpha}{n + \alpha - 1}, \quad \lim_{n \to \infty} x_n = \alpha \)

\[ q_n(\eta) = \frac{\Gamma(\alpha)}{\Gamma(n + \alpha)} \frac{\Gamma(n + \alpha \eta)}{\Gamma(\alpha \eta)} = \frac{(\alpha \eta)_n}{(\alpha)_n} \]

\( q_0(\eta) = 1 \quad q_1(\eta) = \eta \)

\[ p_k^n(\eta) = \frac{x_n!}{x_{n-k}!x_k!} q_k(\eta) q_{n-k}(1 - \eta) \]
\[
p_k^{(n)}(\eta) = \binom{n}{k} \frac{\Gamma(\alpha)}{\Gamma(\eta \alpha) \Gamma((1 - \eta) \alpha)} \frac{\Gamma(\eta \alpha + k) \Gamma((1 - \eta) \alpha + n - k)}{\Gamma(\alpha + n)}
\]

**Pólya-Markov distribution**

G. Pólya (1923): urn scheme. From a set of \(b\) blue balls and \(r\) red balls contained in an urn one extracts one ball and return it to the urn together with \(c\) balls of the same color. The probability to select in the urn \(k\) blue balls after the \(n\)-th trial is given by \(p_k^{(n)}(\eta)\) with

\[
\eta = \frac{b}{b + r} \quad \alpha = \frac{b + r}{c}
\]

(special cases: \(c = 0; \quad c = -1\))
\( q_1(\eta) = \eta \)
\( q_2(\eta) = \frac{1}{6} \eta (1 + 5\eta) \)
\( q_3(\eta) = \frac{1}{42} \eta (1 + 5\eta)(2 + 5\eta) \)
\( q_4(\eta) = \frac{1}{336} \eta (1 + 5\eta)(2 + 5\eta)(3 + 5\eta) \)
\( q_5(\eta) = \frac{1}{3024} \eta (1 + 5\eta)(2 + 5\eta)(3 + 5\eta)(4 + 5\eta) \)

\( \alpha = 5 \)
asymptotic behavior at large $n$

$$p^n_{k=nx} \sim \frac{1}{n} \frac{(x)_{\alpha\eta-1}(1-x)_{\alpha(1-\eta)-1}}{\Gamma[\alpha\eta] \Gamma[\alpha(1-\eta)]} \Gamma[\alpha]$$

$$x \in [0, 1]$$

$$\sum_{k=0}^{n} p^{(n)}_k \rightarrow n \int_0^1 dx \, p^{(n)}_{k=nx} = 1$$

limiting distribution after centering

$$\frac{p^n_{nx}}{p^n_{n/2}} \sim 2^{\alpha-2} x^{\frac{1}{2}(\alpha-2)} (1-x)^{\frac{\alpha}{2}-1} \rightarrow x \rightarrow y+1/2 \quad (1 - 4y^2)^{\frac{1}{2}(\alpha-2)}$$

Wigner law $\rightarrow$ q-Gaussian ($q=(\alpha-4)/(\alpha-3)$ and $\beta=2(\alpha-2)$)
Leibniz triangle rule is strictly obeyed

\[
\langle k \rangle_n = n \eta
\]

\[
\sum_{k=0}^{n} p_{k}^{(n)}(\eta) = \sum_{k=0}^{n} \binom{n}{k} \varpi_k^n = 1
\]

\[
\varpi_k^{n-1} = \varpi_k^n + \varpi_{k+1}^n
\]
Boltzmann-Gibbs and $S_q$ entropies

$$S_{BG} = - \sum_{k=0}^{n} \binom{n}{k} \frac{p_k^{(n)}}{\binom{n}{k}} \log \frac{p_k^{(n)}}{\binom{n}{k}}$$

$$S_q^{(n)} = \frac{1 - \sum_{k=0}^{n} \binom{n}{k} \left( \frac{p_k^{(n)}}{\binom{n}{k}} \right)^q}{q - 1}$$

$\alpha = 3, \ \eta = 1/2$

$$S_{BG} \sim n \left[ \psi(\alpha) - \eta\psi(\alpha\eta) - (1 - \eta)\psi(\alpha(1 - \eta)) - \frac{1}{\alpha} \right]$$

$$S_{BG} \sim 0.552961 \ n \quad (\alpha = 3; \ \eta = 1/2)$$

$$S_R \sim n \ln 2$$
example 2 - Abel-type polynomials

\[ \mathcal{N}(t) = e^{-\alpha W(-t/\alpha)}, \quad \alpha > 0 \]

\[ W(t)e^{W(t)} = t \quad \text{W-Lambert's function} \]

\[ x_n! = n! \frac{\alpha^{n-1}}{(n + \alpha)^{n-1}} \]

\[ x_n = \frac{n\alpha}{n + \alpha} \left( 1 - \frac{1}{n + \alpha} \right)^{n-2} \quad \lim_{n \to \infty} x_n = \alpha/e \]

\[ q_n(\eta) = \eta \frac{(\eta + \frac{n}{\alpha})^{n-1}}{(1 + \frac{n}{\alpha})^{n-1}} \quad q_0(\eta) = 1, \ q_1(\eta) = \eta \]

\[ p^{(n)}_k(\eta) = \binom{n}{k} \eta(1 - \eta) \frac{(\eta + k/\alpha)^{k-1}(1 - \eta + (n - k)/\alpha)^{n-k-1}}{(1 + n/\alpha)^{n-1}} \]
asymptotic behavior at large $n$

$$\mathcal{P}_{k=nx}^{(n)} \sim \frac{1}{n^{3/2}} \frac{\alpha \eta (1 - \eta)}{\sqrt{2\pi} x^{3/2} (1 - x)^{3/2}}$$

$$\sum_{k=0}^{n} \to n \int_{0}^{1} dx \sim_{\text{large } n} n \frac{1}{n^{1/2}} \frac{\alpha \eta (1 - \eta)}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} dx \frac{dx}{x^{3/2} (1 - x)^{3/2}}$$

$$\sim_{\text{large } n} \lim_{\epsilon \to 0} \frac{4\alpha \eta (1 - \eta)}{\sqrt{2\pi}} \frac{1}{\sqrt{n\epsilon}}$$

$$\epsilon = \frac{A}{n} \quad A = \frac{8(\alpha \eta (1 - \eta))^2}{\pi} \quad \text{(large } n) \quad n \int_{0}^{1} dx \mathcal{P}_{k=nx}^{(n)} = 1$$

limiting distribution after centering

(large $n$) $$\frac{\mathcal{P}_{nx}^{(n)}}{\mathcal{P}_{n/2}^{(n)}} \sim \frac{1}{8[x(1 - x)]^{3/2}} \to_{x \to y+1/2} \frac{1}{(1 - 4y^2)^{3/2}}$$

$$q = 5/3 \quad \beta = -6$$
Leibniz triangle rule is asymptotically obeyed, large $n$

\[ \sum_{k=0}^{n} p_{n}^{(k)}(\eta) = \sum_{k=0}^{n} \binom{n}{k} \omega_k^n = 1 \]

(large $n$) \[ \omega_k^{n-1} \approx \omega_k^n + \omega_{k+1}^n \]
\[ S_{BG} = - \sum_{k=0}^{n} \binom{n}{k} \frac{p_k^{(n)}}{\binom{n}{k}} \log \frac{p_k^{(n)}}{\binom{n}{k}} \sim 2\sqrt{2\pi} \alpha \eta (1 - \eta) \sqrt{n} \]

\[ \alpha = 5 \]
\[ \eta = 0.3 \]

\[ 2\alpha \sqrt{2\pi} \eta (1 - \eta) \sqrt{n} \approx 5.26392 \sqrt{n} \]
\[ S_{BG} = - \sum_{k=0}^{n} \binom{n}{k} \frac{p_k^{(n)}}{\binom{n}{k}} \log \frac{p_k^{(n)}}{\binom{n}{k}} \sim 2\sqrt{2\pi} \alpha \eta (1 - \eta) \sqrt{n} \]

\[ \alpha = 5 \]

\[ \eta = 0.3 \]
Rényi entropy

\[ S_{Re; q} = \frac{1}{1 - q} \log \left[ \sum_{k=0}^{n} \binom{n}{k} \left( \frac{p_k^{(n)}}{n^k} \right)^q \right] \sim n \log 2 \]

\( (\alpha = 3, \eta = 1/2, q = 1/2) \)
final comments

- deformation of the binomial distribution (correlations) with a probabilistic interpretation
- binomial $\rightarrow$ BG and Rényi extensives
- Laplace-de Finetti $\rightarrow$ BG and Rényi extensives
- q-exponential case $\rightarrow$ BG and Rényi extensives
- Abel-type case $\rightarrow$ BG non-extensive; Rényi extensive
- correlations can change the extensive entropy – numerical and analytical calculations
- for a given system there is more than one entropic form, function of the probabilities of the microscopic states, which are extensive
- how can we choose the correct entropic form to associate with the thermodynamical entropy of a system?
references


