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Non-extensive random walks

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Abstract

Stochastic variables whose addition leads to q-Gaussian distributions $G_q(x) \propto [1+(q-1)\beta x^2]_+^{1/(1-q)}$ (with $\beta>0$, $1\leqslant q<3$ and where $[f(x)]_+ = max\{f(x), 0\}$) as limit law for a large number of terms are investigated. Random walk sequences related to this problem possess a simple additive–multiplicative structure commonly found in several contexts, thus justifying the ubiquity of those distributions. A characterization of the statistical properties of the random walk step lengths is performed. Moreover, a connection with non-linear stochastic processes is exhibited. q-Gaussian distributions have special relevance within the framework of non-extensive statistical mechanics, a generalization of the standard Boltzmann–Gibbs formalism, introduced by Tsallis over one decade ago. Therefore, the present findings may give insights on the domain of applicability of such generalization.

Keywords: Non-extensivity; Random-walks; Anomalous diffusion

1. Introduction

Throughout the last decade increasing attention is being given to "non-extensive statistical mechanics" (NSM) [1], a theory that extends the standard Boltzmann–Gibbs one as an attempt to embrace meta-equilibrium or out-of-equilibrium regimes.

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The cornerstone of the non-extensive formalism is the entropy $S_q = k(1 - \int \mathrm{d}x\,[\rho(x)]^q)/(q-1)$ [2], where k is a positive constant and ρ a normalized probability density function (PDF). S_q is an increasing function of Rényi entropy $S_q^R = k \ln \int \mathrm{d}x [\rho(x)]^q/(1-q)$ [3], the standard entropy being recovered in the limit $q \to 1$. Optimization of any of these two generalized entropies, under the constraints of constant norm and q-generalized second moment [4], leads to q-Gaussian PDFs defined as

$$G_q(x) = \bar{G}_q[1 + (q-1)\beta x^2]_+^{1/(1-q)}\,, \tag{1}$$

where $\beta > 0$, $q < 3$, $\bar{G}_q$ is a normalization constant, and the subindex + indicates that $G_q(x) = 0$ if the expression between brackets is non-positive. The usual Gaussian distribution is recovered in the $q \to 1$ limit. The PDF defined by Eq. (1) also contains the Cauchy ($q = 2$) and Student's t-distributions ($q = (n+3)/(n+1)$, for n degrees of freedom).

The extensive research about this proposal brought to the surface the fact that many phenomena can be well described by q-Gaussian PDFs, from Bose–Einstein condensates [5] or turbulent flows [6] in physics to stock returns [7] in finances (see Ref. [1] for many other examples). Despite the apparent ubiquity of q-Gaussian PDFs, giving indirect support to the applicability of NSM, a comprehensive description in terms of first principles has still to be worked out. Nevertheless, the fact that q-Gaussian PDFs are so frequently found leads one to think that a simple statistical mechanism, as for other standard limit distributions, may be behind. Then, assuming that sums of certain random variables fall within the basin of attraction of a given q-Gaussian PDF, a basic question arises: Which is the nature of such random variables? An answer to this question may give useful insights on the general framework of applicability of NSM.

While (i) properly scaled sums of independent variables with finite second moment (identically distributed or not, as far as the Lindeberg condition applies) converge to the Gaussian law [8] and (ii) sums of independent variables with divergent second moment may attain a Lévy limit [9], the kind of stochastic variables that concern NSM are expected to have some sort of strong dependence. This idea is consistent with the nature of the systems for which NSM is supposed to apply, e.g., systems with long-range interactions or long-term memory. The particular kind of dependence involved may be investigated in terms of one-dimensional random walks, where the position of a walker in state space, at each instant, is given by the summation of random variables, the individual step lengths. As a counterpart, one can also investigate the associated evolution equation for the probability density of the walker position.

The paper is organized as follows. By starting from the non-linear diffusion equation, and inter-relating previous results in the literature, we will find in Section 2 generalized random-walkers with additive–multiplicative structure whose associated time-dependent PDFs are q-Gaussian. Stochastic properties of the step-lengths for purely diffusive spreading are analyzed in Section 3, while Section 4 focuses on additive–multiplicative processes with time-independent coefficients. A summary is presented in Section 5.

2. Generalized random-walks

While the generalization of the diffusion equation with spatial derivative of fractional order leads to Lévy solutions [10], the non-linear generalization

$$\partial_t \rho = \tfrac{1}{2} D \partial^2_{xx} \rho^\nu \,, \tag{2}$$

with $\nu, D \in \Re$, for $\rho(x, t=0) = \delta(x)$ and natural boundary conditions, has a time-dependent solution of the Barenblatt–Pattle type [11], namely,

$$\rho_\nu(x,t) = g_\nu(t) \left[1 + (1-\nu) \frac{x^2}{a_\nu^2(t)} \right]_+^{1/(\nu-1)} \,, \tag{3}$$

where, $g_\nu(t) = 1/[\gamma_\nu a_\nu(t)]$, $a_\nu(t) = [\nu(\nu+1)\gamma_\nu^{1-\nu} Dt]^{1/(\nu+1)}$, being

$$\gamma_\nu = \begin{cases} \dfrac{\Gamma(\nu/(\nu-1))}{\Gamma((3\nu-1)/(2(\nu-1)))} \sqrt{\dfrac{\pi}{\nu-1}} & \text{if } \nu > 1 \,, \\ \sqrt{\pi} & \text{if } \nu = 1 \,, \\ \dfrac{\Gamma((\nu+1)/(2(1-\nu)))}{\Gamma(1/(1-\nu))} \sqrt{\dfrac{\pi}{1-\nu}} & \text{if } -1 < \nu < 1 \,. \end{cases} \tag{4}$$

The $(2-\nu)$-Gaussian PDF defined by Eq. (3) is normalizable for $\nu > -1$ and has finite second moment for $\nu > \frac{1}{3}$. For $\nu > (<) 0$, it must be $D > (<) 0$ (see Ref. [12] for details). The Cauchy distribution corresponds to $\nu = 0$, in which case there is no diffusion. However, in the limit $\nu \to 0$, diffusion can be restored by making $D \to \infty$ in such a way that $D\nu > 0$ remains finite. The scaling of x with time in Eq. (3) indicates superdiffusive behavior for $-1 < \nu < 1$, normal diffusion for $\nu = 1$ and subdiffusion for $\nu > 1$, although the second moment is finite only for $\nu > \frac{1}{3}$.

Non-linear diffusion describes a variety of transport processes, such as percolation of gases through porous media [13], thin liquid films spreading under gravity [14] or spatial diffusion of biological populations [15]. Eq. (2) can be associated with a stochastic differential equation with multiplicative noise, complemented by the Itô prescription, where the noise amplitude is controlled by a function of the density ρ_ν that in turn satisfies Eq. (2) [12] (see also Ref. [16]):

$$\dot{x} = \sqrt{|D|} [\rho_\nu(x,t)]^{(\nu-1)/2} \eta(t) \,, \tag{5}$$

with $\{\eta\}$ a zero-mean δ-correlated Gaussian process. Accordingly, the non-linear Fokker–Planck equation (FPE) can be derived from a master equation where the transition probabilities depend on $\rho_\nu^{(\nu-1)/2}$ [17]. In terms of transition rates, this means that the probability of transition from one state to another is determined by the probability of occupation of the first state. Notice that, in the limit $\nu \to 1$, one recovers the standard Gaussian diffusion, with constant diffusion coefficient and transition rates.

Through discretization of time in Eq. (5), one obtains a random walker whose position x_n at time $t = n\tau > 0$ evolves according to

$$x_{n+1} = x_n + \sqrt{|D|\tau} [\rho_\nu(x_n, n\tau)]^{(\nu-1)/2} \xi_n \,, \tag{6}$$

where the stochastic process $\{\xi\}$, obtained from $\xi_n = 1/\sqrt{\tau}\int_{n\tau}^{(n+1)\tau} \eta(t)\,\mathrm{d}t$, is Gaussian with $\langle \xi_i \rangle = 0$ and $\langle \xi_i \xi_j \rangle = \delta_{ij}$ [18]. Eq. (5) represents a one-dimensional random walker with a sort of statistical feedback in which the steps depend on the local density of an ensemble of identical walkers. That is, they interact via a sort of mean-field. Fig. 1 shows the PDFs associated with the random walkers defined by Eq. (6), for different values of v, in agreement with the continuous PDF given by Eq. (3). Consequently, collapse of the PDFs would be observed by scaling x_n with $n^{1/(v+1)}$.

Let us rewrite Eq. (5) by substitution of the explicit expression for $\rho_v(x,t)$ given by Eq. (3), which is the solution for a localized initial distribution. Then, one obtains

$$\dot{x} = A_v(t)\left[1 + (1-v)\frac{x^2}{a_v^2(t)}\right]_+^{1/2} \eta(t)\,, \tag{7}$$

with $A_v(t) = \sqrt{|D|}[g_v(t)]^{(v-1)/2} \propto t^{(1-v)/(2(v+1))}$.

When $v \in (-1, 1]$, the expression between brackets is always positive; thus, there is no cut-off and the condition expressed by the subindex "+" can be suppressed. From here on, we will focus on this instance, corresponding to long-tailed PDFs, the case of interest in most applications. For this regime, it can be straightforwardly shown by standard methods [18] that Eq. (7) is equivalent to the *linear* (Itô) stochastic equation

$$\dot{x} = A_v(t)\eta_A(t) + xM_v(t)\eta_M(t)\,, \tag{8}$$

where $M_v(t) = \sqrt{(1-v)}A_v(t)/a_v(t) \propto t^{-1/2}$, and η_A, η_M are two independent zero-mean δ-correlated Gaussian noise sources with $\langle \eta_X(t)\eta_X(t')\rangle = \delta(t-t')$, for $X = A, M$.

Additive–multiplicative processes have been considered in the literature to model a variety of phenomena, as the evolution of the distance from an invariant manifold in on–off intermittency [19], laser intensity fluctuations [20], oscillators with fluctuating parameters [21], economic models [22], amongst others [23,24]. While in most cases either one or the other source has a dominant role, there are also regimes where both

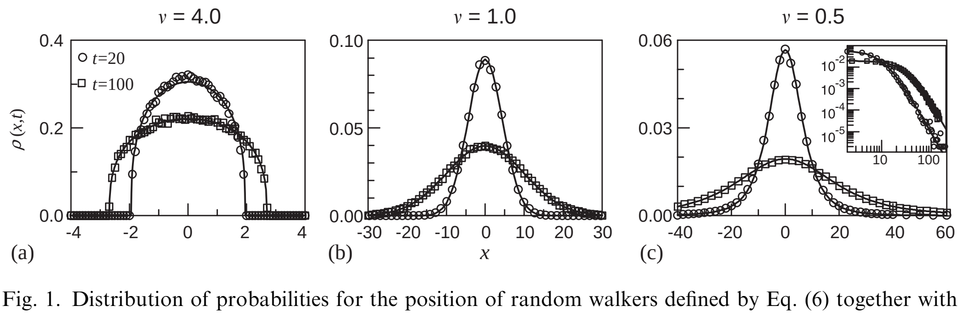

Fig. 1. Distribution of probabilities for the position of random walkers defined by Eq. (6) together with Eq. (3), at two times indicated in the figure. (a) $v = 4.0$ (subdiffusion), (b) $v = 1$ (normal diffusion) and (c) $v = 0.5$ (superdiffusion). In all cases, $D = 1$ and $\tau = 10^{-1}$. Symbols correspond to the histogram built from 10^5 walkers, starting at $x_1 = 0$ and solid lines to the analytical expression given by Eq. (3). For $v > 1$ a cut-off occurs. Inset in (c): log–log scale to watch the power-law tails.

noises give similar contributions. Notice that, instead of two uncorrelated noises in Eq. (8), one might consider a unique source with an appropriate time lag, such the two processes become uncorrelated. Eq. (7) can be interpreted as arising from a unique effective noise amplitude embodying the additive and multiplicative contributions to the linear Eq. (8). In fact, both Eqs. (7) and (8) yield the same diffusion equation

$$\partial_t \rho = \tfrac{1}{2}\partial^2_{xx}\{[A^2_\nu(t) + M^2_\nu(t)x^2]\rho\} \ . \tag{9}$$

Moreover, both this linear diffusion equation and the non-linear Eq. (2) give the same evolution for the localized initial condition $\rho(x,0) = \delta(x)$ (see also Ref. [25]). In both cases, the spreading is ruled by a spatially inhomogeneous diffusion coefficient with symmetry around $x = 0$.

The discretized version of Eq. (8), for $n>0$, reads

$$x_{n+1} = A_{\nu,n}\xi_{A,n} + (1 + M_{\nu,n}\xi_{M,n})x_n \ , \tag{10}$$

where

$$\begin{aligned} A_{\nu,n} &= \sqrt{|D|\tau}[\gamma^2_\nu(\nu+1)\nu D\tau n]^{(1-\nu)/(2(\nu+1))} \equiv \alpha_\nu n^{(1-\nu)/(2(\nu+1))} \ , \\ M_{\nu,n} &= \sqrt{(1-\nu)/[|\nu|(\nu+1)n]} \equiv \sqrt{\mu/n} \ , \end{aligned} \tag{11}$$

and the independent processes $\{\xi_A\}$ and $\{\xi_M\}$ are zero-mean δ-correlated Gaussian ones such that $\langle \xi_{A,i}\xi_{A,j}\rangle = \langle \xi_{M,i}\xi_{M,j}\rangle = \delta_{ij}$; as comes out after taking $\xi_{X,n} = 1/\sqrt{\tau}\int_{n\tau}^{(n+1)\tau}\eta_X(t)\,\mathrm{d}t$ and $X_{\nu,n} = X_\nu(n\tau)\sqrt{\tau}$, with $X = A, M$. The rightmost identities in Eq. (11) define α_ν and $\mu \equiv \mu_\nu$, respectively. Noise amplitudes depend on n: while $A_{\nu,n}$ increases with n, $M_{\nu,n}$ decreases. Eq. (8) is singular at $n = 0$; however, the limit $t \to 0$ can be approached by taking small enough τ. Although the time dependence gets a simpler form through appropriate scaling, we will keep the original form explicitly. In the particular case $\nu = 1$, one recovers the purely additive random process, with a time-independent coefficient α_1, i.e., $x_{n+1} = x_n + \alpha_1\xi_n$.

For $-1<\nu\leqslant 1$, the walkers given by Eqs. (6) and (10) are equivalent in law, as expected from the equivalence of their associated Itô–Langevin equations (ILEs) for initial conditions localized at the origin. For instance, Eq. (10), for $\nu = 0.5$ and starting at $x_1 = 0$, yields the same PDFs shown in Fig. 1(c). As for non-linear diffusion, the PDFs of additive–multiplicative walkers are stable through scaling by $n^{1/(\nu+1)}$. Of course, for arbitrary initial conditions, both processes give different time evolution.

3. Statistical properties of non-extensive walkers

The linear character of expression (10) allows to obtain, by recurrence, an explicit formula for the position of the walker, starting at $x_1 = 0$, in terms of the random sequences $\{\xi_A\}$ and $\{\xi_M\}$, namely,

$$x_{n+1} = \sum_{j=1}^{n} A_{\nu,j}\xi_{A,j}\prod_{i=j+1}^{n}(1 + M_{\nu,i}\xi_{M,i}) \ . \tag{12}$$

In the case $\nu>1$, non-linearity subsists due to the cut-off condition and, therefore, it is not possible to obtain a closed form expression as above. When $\nu=1$, the product equals one (since $M_{1,i}=0$ for all i), $A_{1,j}=\alpha_1$ (for all j), and x_n scaled by $n^{1/2}$ tends in distribution to a stable normal distribution, in agreement with the standard central limit theorem. However, as we have seen, in the cases $\nu\neq 1$, *the distribution of the sum* x_n, *scaled by* $n^{1/(\nu+1)}$, *tends to a stable* $(2-\nu)$-*Gaussian.*

The position of the walker at time $n\tau$, x_n, can also be written as the sum $x_n=\sum_{j=1}^{n-1} s_j$, where the step length at time $j\tau$,

$$s_j \equiv x_{j+1}-x_j = A_{\nu,j}\xi_{A,j} + M_{\nu,j}\xi_{M,j}x_j \,, \tag{13}$$

has a sort of noisy memory of the full history of the walker, through x_j. Recalling that the variables $\{\xi_A\}$ and $\{\xi_M\}$ are uncorrelated, and noticing that $\langle s_j\rangle=0$, then, the two-time correlation of the increments is

$$\langle s_j s_{j'}\rangle = \delta_{jj'}(A^2_{\nu,j} + M^2_{\nu,j}\langle x_j^2\rangle)\,, \tag{14}$$

where, from Eqs. (11) and (12),

$$\langle x_j^2\rangle = \alpha_\nu^2 \frac{\Gamma(j+\mu)}{(j-1)!}\sum_{k=1}^{j-1}\frac{k!\,k^{(1-\nu)/(\nu+1)}}{\Gamma(k+\mu+1)}\,.$$

By means of the Stirling approximation, for large j and $\nu>\frac{1}{3}$, one obtains $\langle x_j^2\rangle = a_\nu^2(j\tau)/(3\nu-1)$. This expression coincides with $\langle x^2\rangle=\int x^2\rho_\nu(x,t)\,\mathrm{d}x = a_\nu^2(t)/(3\nu-1)$, where $t=j\tau$. For $\nu\leqslant\frac{1}{3}$, the Stirling formula yields $\langle x_j^2\rangle=\phi(\nu)\,a_\nu^2(j\tau)j^{(1-\nu-2|\nu|)/(|\nu|(\nu+1))}$, where $\phi(\nu)$ is a function of ν only, that goes to 1 when $\nu(\nu+1)\to 0$ and diverges at $\nu=\frac{1}{3}$. This result is compatible with the fact that $\langle x^2\rangle$ is divergent in this case, as there is an additional increase of $\langle x_j^2\rangle/a_\nu^2(j\tau)$ given by the factor with positive power of j, although the variance of a finite data series is finite. The divergence is logarithmic at the marginal case $\nu=\frac{1}{3}$.

The increments s_j are δ-correlated, in accord with a Hurst exponent $H=\frac{1}{2}$ previously found [12]. For large j, Eq. (14) leads to

$$\langle s_j^2\rangle = \frac{2a_\nu^2(j\tau)}{(\nu+1)(3\nu-1)j} \sim j^{(1-\nu)/(1+\nu)} \tag{15}$$

for $\nu>\frac{1}{3}$, and a divergent second moment for $\nu\leqslant\frac{1}{3}$. It is easy to verify, in the former case, that $\langle x_n^2\rangle=\sum_{j=1}^{n-1}\langle s_j^2\rangle$, as it must be for a sum of uncorrelated variables. Moreover, $max_j[\langle s_j^2\rangle/\langle x_n^2\rangle]\sim 1/n$; thus, none of the steps gives a predominant contribution to the dispersion of the sum. However, higher-order correlations of the steps are not generically null. For instance,

$$\langle s_j^2 s_{j'}^2\rangle - \langle s_j^2\rangle\langle s_{j'}^2\rangle \neq 0\,. \tag{16}$$

Therefore, the steps are not mutually independent, although uncorrelated (that is, with vanishing linear correlation). The lack of linear correlation together with non-null higher-order correlations are also observed, for instance, in GARCH processes [26]. As a consequence of their statistical properties, the sums here considered do not converge in distribution to the Gauss or Lévy limiting laws.

4. Random additive–multiplicative processes with constant coefficients

Inclusion of a linear drift, both in the non-linear diffusion equation as well as in the linear equation with quadratic diffusion coefficient, leads, for arbitrary initial conditions, to a steady state of the q-Gaussian form, as soon as the coefficients attain steady values. In particular, if the coefficients are time-independent, as in

$$\partial_t \rho = \gamma \partial_x (x\rho) + \tfrac{1}{2}\partial^2_{xx}([A^2 + M^2 x^2]\rho)\,, \tag{17}$$

with $\gamma \in \Re$ and A, M positive constants, then, the stationary solution is

$$\rho^s_\nu(x) = \frac{1}{\beta_\nu \gamma_\nu}\left[1 + (1-\nu)\frac{x^2}{\beta_\nu^2}\right]^{1/(\nu-1)}\,, \tag{18}$$

with $\beta_\nu = \sqrt{1-\nu}\,A/M$ and $\nu = \gamma/(\gamma + M^2)$. FPE (17) can also be cast in the form $\partial_t \rho = \gamma \partial_x (x\rho) + \frac{1}{2} D\partial^2_{xx}([\rho^s_\nu(x)]^{\nu-1}\rho)$, with $D = A^2[\beta_\nu \gamma_\nu]^{1-\nu}$, putting into evidence the connection between diffusion coefficient and long-time solution.

From the same considerations made in Section 2 (see also Refs. [27,28]), ILEs associated to Eq. (17), for $\nu \leqslant 1$, are

$$\dot{x} = -\gamma x + [A^2 + M^2 x^2]^{1/2}\eta(t)\,, \tag{19}$$

$$\dot{x} = -\gamma x + A\eta_A(t) + xM\eta_M(t)\,, \tag{20}$$

where the random processes $\{\eta\}$, $\{\eta_A\}$, $\{\eta_M\}$ are those defined in Section 2.

A relevant quantity for this problem is $\varepsilon \equiv M^2/\gamma$, representing the relative strength of the drift fluctuations. Then, ν can be expressed as $\nu = 1/(1+\varepsilon)$ and, in order to obtain $\nu \in (-1, 1]$, it must be either $\varepsilon \geqslant 0$ or $\varepsilon < -2$. Note that steady solutions are possible even under repulsive drift ($\gamma < 0$), as soon as the amplitude of the multiplicative noise is relatively large, i.e., such that $\varepsilon < -2$. For steady solutions with finite second moment ($\nu > \frac{1}{3}$), the relative amplitude of the multiplicative noise can not be arbitrarily large, namely, $0 \leqslant \varepsilon < 2$ must hold.

Time discretization in Eq. (20), for $n \geqslant 0$, leads to

$$x_{n+1} = \bar{A}\xi_{A,n} + (1 - \kappa + \bar{M}\xi_{M,n})x_n\,, \tag{21}$$

where the processes $\{\xi_A\}$ and $\{\xi_M\}$ are defined as in Section 2, $\kappa = \gamma\tau \ll 1$, $\bar{A} = A\sqrt{\tau}$ and $\bar{M} = M\sqrt{\tau}$. By recurrence, and taking $x_0 = 0$, the walker can be solved as

$$x_{n+1} = \bar{A}\sum_{j=0}^{n} \xi_{A,j} \prod_{i=j+1}^{n} (1 - \kappa + \bar{M}\xi_{M,i})\,. \tag{22}$$

The variance at time $n\tau$ is

$$\langle x_n^2 \rangle = \bar{A}^2 \sum_{j=0}^{n-1} \prod_{i=j+1}^{n-1} \Lambda = \bar{A}^2\,\frac{\Lambda^n - 1}{\Lambda - 1}\,, \tag{23}$$

where $\Lambda = (1-\kappa)^2 + \bar{M}^2$. For $|\kappa| \ll 1$, $\Lambda \simeq 1 + \kappa(1-3v)/v$; therefore $\Lambda < 1$, for $v > \frac{1}{3}$. Then, in the limit of large n, $\langle x_n^2 \rangle$ coincides with $\int x^2 \rho_v^{(s)}(x)\,dx = \beta_v^2/(3v-1) = \bar{A}^2/(2\kappa - \bar{M}^2)$, for $v > \frac{1}{3}$, and increases with n otherwise (because $\Lambda \geqslant 1$).

Since the stationary distribution of the process is a q-Gaussian, this PDF represents the limit law for the random walk with constant coefficients given by Eq. (21). That is, the $(2-v)$-Gaussian is the stable distribution of x_n (without further scaling), for sufficiently large n, as illustrated in Fig. 2.

In the particular limit of null average drift ($\varepsilon \to \pm\infty$, hence $v \to 0^{\pm}$), the steady state is the Cauchy distribution. In general, $A/\sqrt{\gamma + M^2} \equiv \bar{A}/\sqrt{\kappa + \bar{M}^2}$ determines the width of the limit distribution [27], while only the quotient $\varepsilon \equiv M^2/\gamma \equiv \bar{M}^2/\kappa$ rules the exponent v and therefore the tail law. In the limit $\varepsilon \to 0$ (hence $v \to 1$), the normal distribution is obtained. However, as soon as the drift is sensitively *noisy*, a q-Gaussian with $q \equiv 2 - v \neq 1$ arises.

Increments $s_j \equiv x_{j+1} - x_j = \bar{A}\xi_{A,j} + (-\kappa + \bar{M}\xi_{M,j})x_j$ have zero mean (taking $x_0 = 0$) and variance

$$\langle s_j^2 \rangle = \bar{A}^2 \frac{\Lambda^j(\kappa^2 + \bar{M}^2) - 2\kappa}{\Lambda - 1} . \tag{24}$$

If $v > \frac{1}{3}$ ($\Lambda < 1$), the variance goes to $2\bar{A}^2\kappa/(2\kappa - \bar{M}^2)$ for large j, while, it increases with j, when $v < \frac{1}{3}$. The linear correlation is

$$\langle s_j s_{j+l} \rangle = \bar{A}^2 \kappa^2 \frac{(1-\kappa)^{l-1}}{\Lambda - 1}(1 + \Lambda^j[1 - k - \bar{M}^2/\kappa]) \quad \text{for } l > 0 . \tag{25}$$

In the particular case $\kappa = 0$, the steps are uncorrelated and the limit law is the Cauchy distribution (2-Gaussian). For $\kappa \neq 0$, the increments s_j are history-dependent and correlations are imposed by the deterministic forcing, even when $v = 1$. Note, however, that, if $\bar{M} = 0$, then Eq. (22) can be expressed as, $x_{n+1} = \bar{A}\sum_{j=0}^{n}$

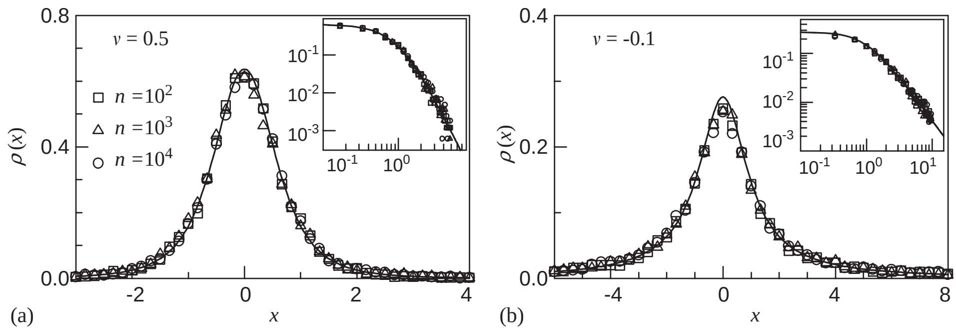

Fig. 2. Distribution of probabilities for the position of random walkers defined by Eq. (21), for three values of the number of steps n, indicated in the figure. Parameters are (a) $(A, M, \gamma, \tau) = (2, 2, 4, 0.01)$, hence $v = 0.5$; (b) $(A, M, \gamma, \tau) = (5, 5, -25/11, 0.01)$, hence $v = -0.1$. Symbols correspond to histograms built from 10^4 random walkers, starting at $x_0 = 0$, and solid lines to Eq. (18). Insets: log–log representation to watch the tails.

$(1-\kappa)^{n-j}\xi_{A,j}$, that is, a sum of independent variables with finite second moment whose limit law is Gaussian. For $v \neq 1$ ($\bar{M} \neq 0$), a similar disentanglement does not seem to be possible. In this case, the steps have a noisy record of the history. Therefore, the subtle presence of fluctuations ($\bar{M} \neq 0$) in the drift parameter κ produces the deviation from the standard behavior. In other words, as soon as the relative amplitude of the drift fluctuations ε becomes significant, q-Gaussian PDFs, with tails ruled by ε, arise.

5. Summary

We have characterized random variables whose addition has a q-Gaussian as limit distribution. We considered (i) a purely diffusive situation, where the density spreads out, and (ii) a case with external linear drift, where a steady density is attained. In case (i), considered in Sections 2 and 3, the increments of the random walks (with time-dependent coefficients) are non-identically distributed random variables with null linear correlation but non-null higher-order correlations. In case (ii), considered in Section 4, q-Gaussian PDFs arise from the combination of an additive process together with a fluctuating linear drift. In both cases (i) and (ii), the increments originate from two sequences of independent, identically distributed Gaussian random variables $\{\xi_A\}$ and $\{\xi_M\}$, the first acting additively and the other through a linear multiplicative contribution. As far as additive–multiplicative structured sequences are frequent in several contexts and its simpler linear form leads to q-Gaussian PDFs, the ubiquity of these distributions cannot be surprising. As a perspective, it would be interesting to investigate a possible connection between additive–multiplicative processes and those underlying the so called "superstatistics" proposed by Beck and Cohen [29], as well, as whether the kind of correlations making S_q additive [30] are related to those found in non-extensive walkers.

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