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Chaos synchronization in long-range coupled map lattices

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Abstract

We investigate the synchronization phenomenon in 1D coupled chaotic map lattices where the couplings decay with distance following a power-law. Depending on the number of maps, the coupling strength and the range of the interactions, complete chaos synchronization may be attained. The synchronization domain in the coupling parameter space can be analytically determined by means of the condition of negativity of the largest transversal Lyapunov exponent. In this Letter we use previously found analytical expressions for the synchronization frontier to analyze in detail the role of all the system parameters in the ability of the lattice to achieve complete synchronization. Analytical predictions are shown to be in accord with the outcomes of numerical experiments.

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Keywords: Synchronization; Coupled map lattices; Long-range interactions

Coupled map lattices (CMLs), dynamical systems with discrete space and time, are being intensively investigated nowadays as models of spatiotemporal phenomena occurring in many systems of physical, biological and technical interest [1]. CMLs may arise, for example, from the discretization of a class of diffusion–reaction systems for which the reactive term is a sequence of delta-functions [2]. Another physical problem which can be dealt using CMLs is a chain of coupled particles under a spatially periodic potential function and impulsive forcing [3]. Whenever possible, the reduction of a given physical problem to the form of a CML is advantageous from the point of view of numerical simulations, since the computer time is substantially reduced, when compared to that required for numerical integration of either partial or ordinary differential equations.

In this Letter we will deal with the phenomenon of synchronization and, in particular, amongst the various kinds of synchronized behavior, with the *complete synchronization* (CS) [4] occurring in CMLs with regular long-range interactions. Most work done so far on synchronization in CMLs has focused on two extreme coupling types: local (nearest-neighbor) [5]

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and global ("mean field") ones [6]. However, nonlocal couplings are relevant to a variety of situations ranging from neural networks [7] to physico-chemical reaction systems [8].

We consider chains of N coupled one-dimensional (1D) chaotic maps $x \mapsto f(x)$ whose evolution is given by [9]

$$x_{n+1}^{(i)} = (1-\varepsilon) f\bigl(x_n^{(i)}\bigr) + \frac{\varepsilon}{\eta} \sum_{r=1}^{N'} \frac{f(x_n^{(i-r)}) + f(x_n^{(i+r)})}{r^{\alpha}}, \qquad (1)$$

where $x_n^{(i)}$ represents the state variable for the site i $(i = 1, 2, \ldots, N)$ at time n, $\varepsilon \geqslant 0$ is the coupling strength, $\alpha \geqslant 0$ controls the effective range of the interactions and $\eta = 2\sum_{r=1}^{N'} r^{-\alpha}$ is a normalization factor, with $N' = (N-1)/2$ for odd N. Boundary conditions are periodic: $x_n^{(i\pm N)} = x_n^{(i)}$, for $i = 1, 2, \ldots, N$, and the minimal intersite distance over the ring is considered to evaluate the power-law couplings. The main interest in this coupling scheme resides in the fact that it allows to investigate the role of the range of the interactions, scanning from the local ($\alpha \to \infty$) to the global ($\alpha = 0$) cases [10].

CS takes place when the dynamical variables which define the state of each map adopt, after a transient, the same value for all the coupled maps at all times n, i.e, $x_n^{(1)} = x_n^{(2)} = \cdots = x_n^{(N)} \equiv x_n^{(*)}$. It can be easily verified that this state is solution of Eq. (1). Depending on the number of maps and on the range of the interactions, there may exist an interval of values of the coupling strength ε, for which such state is spontaneously attained, as we have analytically shown in a previous work [11]. It is our purpose here to scrutinize the role of all the parameters in the ability of the system to synchronize. Analytical results will be compared with the outputs of numerical simulations performed for diverse one-dimensional chaotic maps $f(x)$ defined on the interval $[0, 1]$ and possessing a single attractor.

CS can be characterized by a complex order parameter introduced by Kuramoto for phases (cyclic variables) [12]. Since in this Letter we will deal with maps defined in the unit interval $[0, 1]$, this definition is slightly modified to $R_n = |\frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{2\pi i x_n^{(j)}}|$ for the order parameter amplitude at time n, where the values of $x_n^{(j)}$ are mapped to the unit circle [9]. Typically a time-averaged amplitude $\bar{R}$ is computed after a time interval long enough to allow the chain attain the asymptotic state. In the CS state, one has $\bar{R} = 1$ within a small allowed deviation.

Another diagnostic of complete synchronization can be obtained from the Lyapunov spectrum (LS) of the CS states. If the chaotic maps are completely synchronized, the maximal Lyapunov exponent, in the direction parallel to the synchronization manifold (SM), is strictly positive. The negativity of the second largest Lyapunov exponent, which belongs to the direction transversal to the SM, indicates the stability of the synchronized state under small transversal displacements [13]. If the SM is the only attractor, then even trajectory points far from the SM are still attracted to it [14].

In our case the lattice dynamics given by Eq. (1) can be written as $x_{n+1}^{(i)} = \sum_j F_{ij} f(x_n^{(j)})$, where $\mathbf{F}$ is a matrix of the form

$$\mathbf{F} = \left[(1-\varepsilon)\mathbb{1} + \frac{\varepsilon}{\eta}\mathbf{B}\right], \qquad (2)$$

with $\mathbb{1}$ the $N \times N$ identity matrix and $\mathbf{B}$ defined by $B_{jk} = (1-\delta_{jk})/r_{jk}^{\alpha}$, being $r_{jk} = \min_{l\in\mathcal{Z}} |j - k + lN|$.

The Lyapunov spectrum is obtained from the dynamics of tangent vectors ξ, which in turn is obtained by differentiation of the original evolution equations. In matrix form the tangent dynamics reads $\xi_n = \mathcal{T}_n \xi_0$, where $\mathcal{T}_n$ is product of n Jacobian matrices calculated at successive points of a given trajectory. If $\Lambda^{(1)}, \ldots, \Lambda^{(N)}$ are the eigenvalues of $\hat{\Lambda} = \lim_{n\to\infty} (\mathcal{T}_n^T \mathcal{T}_n)^{\frac{1}{2n}}$, the Lyapunov exponents are obtained as $\lambda^{(k)} = \ln \Lambda^{(k)}$, for $k = 1, \ldots, N$ [15]. Evaluating the Jacobian matrices along the synchronized trajectories, one arrives at the following expression for the Lyapunov spectrum [11]

$$\bar{\lambda}^{(k)} = \lambda_U + \ln\left|1 - \varepsilon + \varepsilon \frac{b^{(k)}}{\eta}\right|, \qquad (3)$$

where $\lambda_U > 0$ is the Lyapunov exponent of the uncoupled chaotic map, and $b^{(k)}$ are the eigenvalues of $\mathbf{B}$ that can be obtained by Fourier diagonalization and, for odd N, read

$$b^{(k)} = 2\sum_{m=1}^{N'} \frac{\cos(2\pi k m/N)}{m^{\alpha}}, \quad 1 \leqslant k \leqslant N. \qquad (4)$$

For even N, summations run up to $N' = N/2$ and half of the N'th term has to be subtracted. The maximal eigenvalue is $b^{(N)} = \eta$ and the minimal one is $b^{(N')}$. Except for the cases $k = N'$, with even N, and $k = N$, the remaining eigenvalues are two-fold degenerate, being $b^{(k)} = b^{(N-k)}$.

In the calculation of Lyapunov exponents $\lambda^{(k)}$, notice that the parameters that define the particular uncoupled map affect only λ_U, while the second term in Eq. (3) is determined by the particular dependence on distance in the regular coupling scheme (a power law in our case). It can be easily verified that, for arbitrary α, the CS state lies along the direction of the eigenvector associated to the largest exponent $\bar{\lambda}^{(N)}$. Therefore, the CS state will be transversally stable if the $(N-1)$ remaining exponents are negative, that is $|1-\varepsilon+\varepsilon b^{(k)}/\eta| < \mathrm{e}^{-\lambda_U}$, $\forall k \neq N$. This is equivalent to requiring that the second largest (or largest transversal) asymptotic exponent, denoted by $\bar{\lambda}^{\perp}$, be negative. This exponent is obtained from Eq. (3) with either $k = 1$ (hence also $k = N-1$ due to degeneracy) or with $k = N'$ (hence also $k = N'+1$ if N is odd), depending on whether $|1-\varepsilon+\varepsilon b^{(1)}/\eta|$ is, respectively, greater or smaller than $|1-\varepsilon+\varepsilon b^{(N')}/\eta|$. The condition $\bar{\lambda}^{\perp} < 0$ leads to $\varepsilon_c < \varepsilon < \varepsilon_c'$ [11] (see also [16]), where

$$\varepsilon_c(\alpha, N, \lambda_U) = \left(1-\mathrm{e}^{-\lambda_U}\right)\left(1-\frac{b^{(1)}}{\eta}\right)^{-1} \quad \text{and} \tag{5}$$

$$\varepsilon_c'(\alpha, N, \lambda_U) = \left(1+\mathrm{e}^{-\lambda_U}\right)\left(1-\frac{b^{(N')}}{\eta}\right)^{-1}. \tag{6}$$

In Fig. 1 we show a variety of critical curves in parameter space (α, ε) obtained for different number of maps. Stable CS states dwell in the region bounded by the $\alpha = 0$ axis, and two curve segments. The critical curves were obtained analytically from Eqs. (5) (lower curve) and (6) (upper curve). The symbols shown stand for the numerical results determined from the condition $\bar{R} = 1$ with a tolerance of 10^{-6}, after a transient of 5×10^3. Two different values of λ_U were considered. Numerical results shown in Fig. 1 were computed for the piecewise linear (a) Bernoulli shift $f(x) = 2x$ (mod 1) (therefore $\lambda_U = \ln 2$) and (b) triangular map [17]

$$f_w(x) = \begin{cases} x/w, & \text{for } 0 \leqslant x \leqslant w, \\ (1-x)/(1-w), & \text{for } w < x \leqslant 1, \end{cases} \tag{7}$$

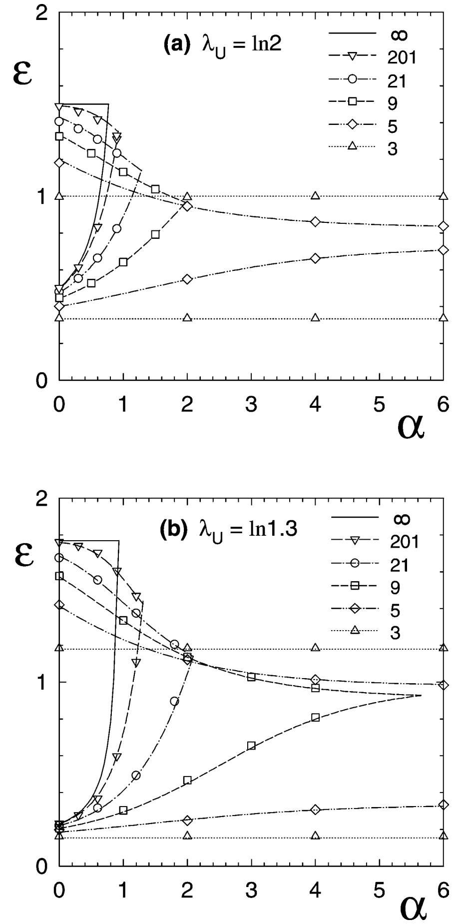

Fig. 1. Synchronization diagram in parameter space (α, ε), for different values of N and $\lambda_U = \ln 2$ (a), $\ln 1.3$ (b). Lines correspond to analytical predictions; symbols to numerical simulations using the Bernoulli (a) or triangular (b) maps. Synchronization is transversally stable in the region between the couple of curves for each set of values of the parameters.

that for $w = 0.074$ yields $\lambda_U \simeq \ln 1.3$ (notice that $\lambda_U = -w\ln w - (1-w)\ln[1-w]$). Additional tests (results not shown here) were performed for other maps such as the logistic map $f(x) = \mu x(1-x)$ with $\mu = 4$ (hence $\lambda_U = \ln 2$) and $\mu = 3.6533$ (hence $\lambda_U \simeq \ln 1.30$) yielding the same degree of agreement. For these interval maps, in principle, one must have $0 \leqslant \varepsilon \leqslant 1$ in order to guarantee that the state variables $x_n^{(j)}$ will remain inside the interval $[0, 1]$. But, reinjection into the interval can be performed through an operation, for instance, (mod 1), such that it does not spoil the Lyapunov exponent of the chaotic uncoupled map. If trajectory points were not reinjected, one can still look at our results as valid for trajectories or trajectory segments as long as the state variables remain confined within the given interval. Anyway, for other maps such as $f(x) = \exp\{-[(x-0.5)/\sigma]^2\}$ [18], one may have any coupling $\varepsilon \geqslant 0$ since the map is naturally defined in the full real axis. Tests performed with this Gaussian map (results not shown in this Letter) are also in good accord with theoretical predictions.

In general terms, we observe that for weak coupling, the maps do not synchronize. As the coupling strength increases, synchronization can occur depending on the system parameters (α, N, λ_U). However, a too high coupling intensity $\varepsilon > \varepsilon'_c$ has a destabilizing influence on the CS state and the chain no longer synchronizes. An upper bound has been also observed previously for other CMLs such as scale-free networks [19], a general class of CMLs [16] and lattices with homogeneous couplings [20]. At first it seems counter-intuitive that a coupling too strong can, in certain cases, be responsible for desynchronization. In order to understand why this effect can occur we have to consider the role of the coupling strength ε in Eq. (1). Small values of ε mean that the dynamics of a given site $x^{(i)}$ is mainly influenced by itself, and weakly by its neighbors. As ε goes to unity, it follows that the contribution from the site itself vanishes, and the dynamics is dictated only by the site neighbors. When ε is further increased, the term $1-\varepsilon$ becomes negative, and the influence of the site itself has a sign opposite to that of its neighbors (the second term in Eq. (1) is always positive), what can eventually destabilize the CS state.

Concerning chain size, Fig. 1 already exhibits the intuitive fact that it is more difficult to synchronize a larger chain than a shorter one, all other parameters being kept fixed. In the limit $N \to \infty$,

$$\varepsilon_c(\alpha, \infty, \lambda_U) = \frac{1 - e^{-\lambda_U}}{1 - C(\alpha)}, \tag{8}$$

where $C(\alpha) = \lim_{N\to\infty} b^{(1)}/\eta$ [11]. This limit is equal to unity for $\alpha > 1$, so that Eq. (8) yields a divergent result. For α outside the domain of convergence of the series, i.e., $\alpha < 1$,

$$C(\alpha) = \frac{1-\alpha}{\pi^{1-\alpha}} \int_0^{\pi} \frac{\cos(x)}{x^{\alpha}}\, dx. \tag{9}$$

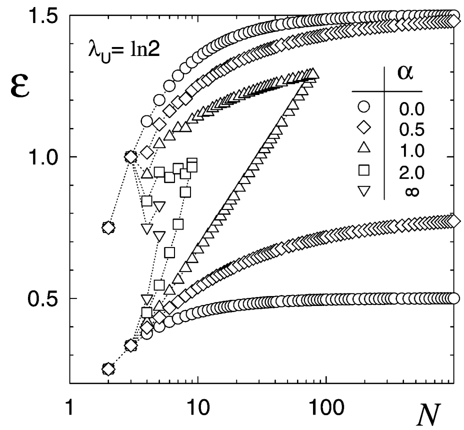

Fig. 2. Synchronization critical lines in the plane (N, ε) for different values of α and $\lambda_U = \ln 2$. In this case $\alpha_c \approx 0.77$. Symbols correspond to theoretical calculations. Dotted lines are guides to the eyes. Synchronization is transversally stable in the region between the two curves for each set of values of the parameters.

In that same range of α one has

$$\varepsilon'_c(\alpha, \infty, \lambda_U) = 1 + e^{-\lambda_U}, \tag{10}$$

which is independent on α [11], thus it yields a straight line in the plots of Fig. 1. From the intersection of $\varepsilon_c(\alpha, \infty, \lambda_U)$ with $\varepsilon'_c(\alpha, \infty, \lambda_U)$ it results a critical value of the interaction range α_c, such that, for $\alpha \leqslant \alpha_c < 1$ ($\alpha_c < d$ in the d-dimensional case [11]), synchronization is possible even in the thermodynamic limit $N \to \infty$ for an appropriate window of ε. Observe the corresponding domains in Fig. 1.

As N diminishes, the upper curve in Fig. 1, which is a straight line for infinite N, gains a negative inclination and extends for large α to values of ε less than unity, indicating that the destabilizing effect of very strong coupling is more easily attained. The lower curve segment has a positive inclination, connecting to the upper curve at a point that forms a cusp for small N. For still smaller N (e.g., $N \leqslant 5$ for $\lambda_U = \ln 2$ and $N \leqslant 8$ for $\lambda_U = \ln 1.3$) the two critical curves do not intersect each other even in the limiting case of first neighbors ($\alpha \to \infty$).

The effect of system size can also be observed in Fig. 2 that exhibits the synchronization domains in the plane (N, ε) for different values of α. Similar plots have been observed in scale-free networks [19], as if there were an average or effective α in such cases. While for $\alpha > \alpha_c$ there is un upper bound $N_b(\alpha, \lambda_U)$ of the number of maps for which synchronization

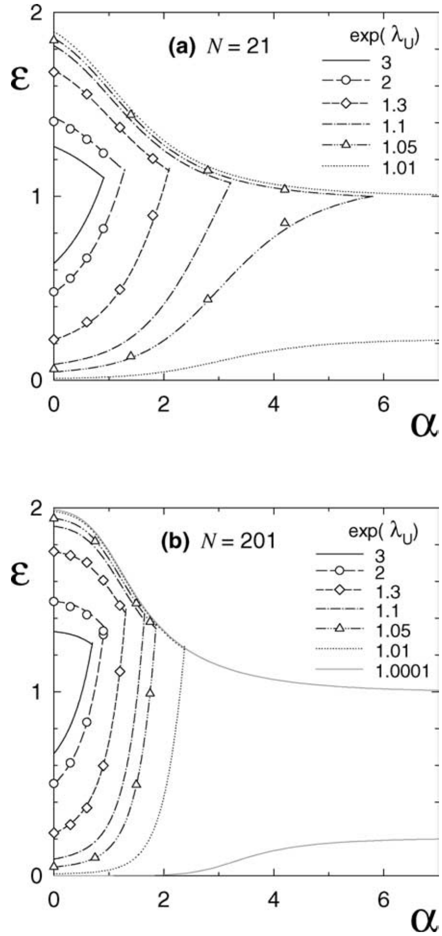

Fig. 3. Synchronization domains in parameter plane (α, ε) for various values of λ_U and $N = 21$ (a), 201 (b) coupled chaotic maps. Numerical simulations were performed for Bernoulli or triangular maps. Critical curves were obtained analytically from Eqs. (5) and (6).

occurs; for $\alpha < \alpha_c$ any number of maps synchronize (because the critical curves in Fig. 2 do not intersect). Generically it is easier to synchronize a small number of maps. Consistently with this observation, chains of small size (e.g., $N \leqslant 5$ for $\lambda_U = \ln 2$) can synchronize for any α, for a certain window of ε that narrows with increasing α. For $N \leqslant 3$ there is, naturally, no dependence on α and the system synchronizes for any $\lambda_U > 0$.

Although Fig. 1(a) and (b) yield qualitative similar results, their comparison makes clear that, as expected, the more chaotic the uncoupled maps are, the more difficult becomes to obtain their synchronization. The influence of the Lyapunov exponent λ_U on the synchronization domains in the parameter space (α, ε) is displayed in Fig. 3. The exhibited numerical results were acquired for $N = 21$ and 201 either Bernoulli or triangular maps.

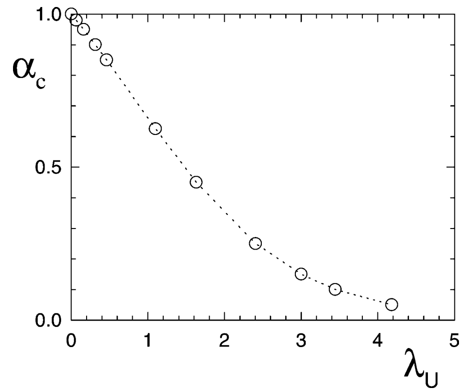

Fig. 4. Critical value α_c, below which synchronization is stable even in the thermodynamic limit, as a function of λ_U (symbols), determined from Eq. (13). The dotted line is a guide to the eyes.

If the positive Lyapunov exponent of the uncoupled map increases, the synchronization domain shrinks, collapsing in the limit $\lambda_U \to \infty$. In the opposite limit of $\lambda_U \to 0^+$ one gets

$$\varepsilon_c \to 0 \quad \text{and} \tag{11}$$

$$\varepsilon_c' \to 2\left(1 - \frac{b^{(N')}}{\eta}\right)^{-1}. \tag{12}$$

This limit value of ε_c' depends on α and N. If $N \to \infty$ and $\alpha \to 0$ (∞), ε_c' goes to 2.0 (1.0) in the limit of vanishing chaos. (All these extreme behaviors are already insinuated in Fig. 3.) As a consequence, the critical value $\alpha_c < 1$, below which the chain synchronizes in the thermodynamic limit, depends on the degree of chaoticity of the uncoupled maps. This dependence can be explicitly obtained by inversion of

$$\lambda_U = \ln\left[\frac{2}{C(\alpha_c)} - 1\right], \tag{13}$$

extracted from Eqs. (8) and (10). The critical value α_c as a function of λ_U is displayed in Fig. 4. In the limit of vanishing (infinite) λ_U, α_c goes to 1.0 (0.0).

When $\alpha > \alpha_c$, it must be $N \leqslant N_b(\alpha, \lambda_U)$ for the system to synchronize, where N_b decreases with increasing $\alpha - \alpha_c$ (as shown in Fig. 2). In the limit $\alpha \to \infty$, it is easy to obtain, from the condition

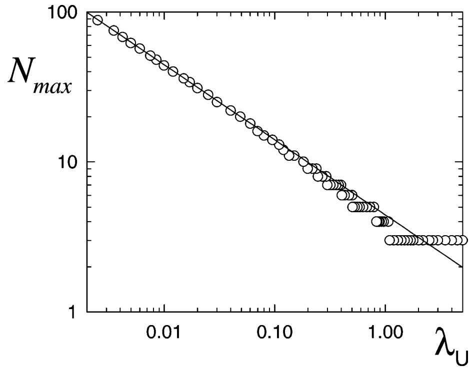

Fig. 5. Maximal number of maps $N_{\max} \equiv N_b(\alpha, \lambda_U)$, for which synchronization can be achieved for any α, as a function of the chaoticity indicator λ_U (circles), determined from the condition $\varepsilon_c' > \varepsilon_c$ for $\alpha = \infty$. The solid line corresponds to the approximation given by Eq. (14).

$\varepsilon_c' > \varepsilon_c$, an approximate expression for the maximal size, valid when one has sufficiently small λ_U and large N_b:

$$N_{\max} \equiv N_b(\infty, \lambda_U) \leqslant \pi \sqrt{\frac{2}{\lambda_U}}. \tag{14}$$

In Fig. 5, we exhibit the maximal size $N_{\max}$ for which synchronization can be achieved in the limit of nearest-neighbor couplings (hence for any α) as a function of λ_U, together with the approximation given by Eq. (14).

Summarizing, we have presented numerical results for the CS states in 1D lattices of coupled identical chaotic maps with interactions that decay with distance as a power law. Those results are in agreement with theoretical predictions derived from the critical lines obtained in previous work from the condition of negativity of the largest transversal Lyapunov exponent [11]. We have scrutinized the role of the system parameters in the ability of the lattice to attain complete synchronization. In numerical simulations we used various chaotic 1D maps with a single attractor. We observed, in the coupling parameter plane, an overall decrease of the area of the synchronization regions, as the number of coupled maps is increased. The shape of these regions is bounded by critical curves which vary with the number of coupled maps in a fashion we were able to predict analytically in excellent agreement with numerical results. We have also studied the dependence of the synchronization regions on the degree of chaoticity of uncoupled maps. In all cases we investigated analytically the behavior of the system under limit values of the parameters. Most of our results could be straightforwardly extended, following [11], to d-dimensional lattices which are expected to exhibit similar qualitative features. For instance, for arbitrary d there is also an upper bound for α_c, which has been shown to be equal to the lattice dimension [11]. Another issue that would be interesting to explore in future works is the influence of perturbations, such as noise or defects, on the synchronization domains.

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