



Vlasov stability of the Hamiltonian mean field model

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Abstract

We investigate the dynamical stability of a fully coupled system of N inertial rotators, the so-called Hamiltonian Mean Field model. In the limit $N \rightarrow \infty$, and after proper scaling of the interactions, the μ -space dynamics is governed by a Vlasov equation. We apply a nonlinear stability test to (i) a selected set of spatially homogeneous solutions of Vlasov equation, qualitatively similar to those observed in the quasi-stationary states arising from fully magnetized initial conditions, and (ii) numerical coarse-grained distributions of the finite- N dynamics. Our results are consistent with previous numerical evidence of the disappearance of the homogeneous quasi-stationary family below a certain energy.

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1. Introduction

Nonextensive statistical mechanics (NSM) was proposed in 1988 [1] as an attempt to account for many phenomena that cannot be described within the standard Boltzmann–Gibbs scenario (see Ref. [2] for reviews). NSM starts from a generalized entropy functional, and using suitable constraints, a generalization of the usual canonical ensemble thermostatics is obtained. Instead of the exponential weight, in NSM one has a q -exponential probability distribution in Γ -space:

$$P(x) \propto \exp_q[-\beta H(x)] \equiv \{1 - \beta(1 - q)H(x)\}^{1/(1-q)} \quad (1)$$

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(if the quantity between curly brackets is negative then $P=0$). Here, x is a point in the N -body phase space of a system described by the Hamiltonian $H(x)$, β a generalized inverse temperature and q the entropic index [2]. Unlike the canonical distribution function [Eq. (1) with $q=1$], which applies to equilibrium situations, the q -canonical distribution (with $q \neq 1$) is believed to be associated to meta-equilibrium or even out-of-equilibrium regimes [2].

The predictions of the theory can in principle be derived from Eq. (1). However, explicit calculations for the anomalous high-dimensional systems of interest for NSM have not been performed yet due to large technical difficulties. For instance, single particle distribution functions require the integration of $P(x)$ over $N-1$ particles.

On the other side, there are many experimental and numerical data well described by q -exponential functions [2]. This agreement is considered as an indirect evidence of the applicability of the NSM to those systems. Moreover, q -exponentials have also been observed in non-Hamiltonian and even non-physical (biological, economical, etc.) systems [3], but in these cases, the connection with Statistical Mechanics (from first principles) is less clear.

A model system potentially important for NSM is the so-called Hamiltonian mean field model (HMF), an inertial XY ferromagnet with infinite-range interactions [4,5]:

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2N} \sum_{i,j=1}^N [1 - \cos(\theta_i - \theta_j)], \quad (2)$$

where the coordinates of each particle are the angle θ_i and its angular momentum p_i . This model can be considered a simple prototype for complex, long-range systems like galaxies and plasmas. In a certain sense, the HMF is a descendant of the mass-sheet gravitational model [5].

The HMF has been extensively studied in the last few years (see Ref. [6] for a review), especially due to its long-lived quasi-equilibrium states, which exhibit breakdown of ergodicity, anomalous diffusion, aging, and non-Maxwell velocity distributions. In some cases q -exponential behaviors were reported: two-time correlation functions [7,8], single particle velocity distributions (truncated q -Gaussians) [9] and temperature relaxation [10]. The simplicity of the HMF, which made possible a full analysis of its statistical properties both in the standard canonical [5] and microcanonical ensembles [11], gives support to the hope that it may be possible to derive explicit first principle predictions, e.g., for single particle velocity distributions.

The most widely investigated quasi-stationary states (QSS) of the HMF arise from special initial conditions (water-bag) that evolve to states with well-defined macroscopic characteristics and whose lifetimes increase with the system size [9]. For a given energy, the QSS is different from the corresponding equilibrium state to which the system eventually evolves. These states have been observed for specific energies $\varepsilon \equiv H/N$ below the critical value $\varepsilon_{\text{cr}}=0.75$ corresponding to a ferromagnetic phase transition. In a magnetization diagram (see Ref. [9] and Fig. 1), a first part of the QSS family lies on the continuation of the high-energy equilibrium line (having zero magnetization) down to the energy $\varepsilon_{\text{min}} \simeq 0.68$. At ε_{min} the $m=0$ line ceases to exist, being substituted by a curve of magnetized states.

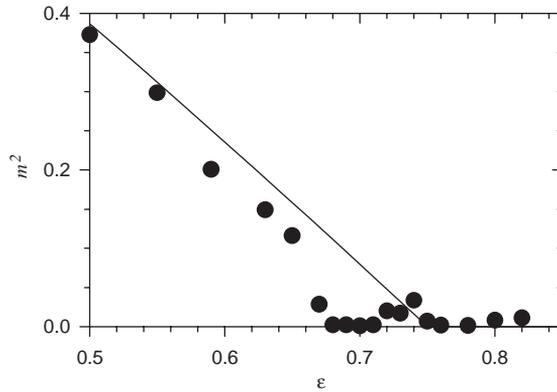


Fig. 1. Squared magnetization versus energy for quasi-stationary states (circles). System size is $N = 10^4$. We averaged over 10 initial conditions and in the time window $500 < t < 2000$ (see Section 4 for numerical details). The full line is the analytical result for thermal equilibrium [5].

This paper is an attempt to understand the origin of such discontinuity. We base our analysis on the associated Vlasov equation, which describes the continuum limit of the Hamiltonian particle dynamics. Using the stability criterion recently developed by Yamaguchi et al. [12], we show that the *coarse-grained* homogeneous (zero-magnetization) QSS family becomes unstable at ϵ_{\min} .

2. Vlasov approach

Introducing radial and tangent unitary vectors $\hat{\mathbf{r}}_i = (\sin \theta_i, \cos \theta_i)$ and $\hat{\boldsymbol{\theta}}_i = (\cos \theta_i, -\sin \theta_i)$, Hamiltonian (2) leads to the equations of motion

$$\begin{aligned} \dot{\theta}_i &= p_i, \\ \dot{p}_i &= \mathbf{m} \cdot \hat{\boldsymbol{\theta}}_i, \end{aligned} \tag{3}$$

for $i = 1, \dots, N$, with the magnetization per particle

$$\mathbf{m} = \frac{1}{N} \sum_{i=1}^N \hat{\mathbf{r}}_i. \tag{4}$$

Eqs. (3) describe the two-dimensional motion of a cloud of N points in the fluctuating mean-field $\mathbf{m}(t)$. In the continuum approximation, the cloud becomes a fluid governed by the Liouville equation for the single-particle probability distribution $f(\theta, p, t)$

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial \theta} + \mathbf{m} \cdot \hat{\boldsymbol{\theta}} \frac{\partial f}{\partial p} = 0 \tag{5}$$

with

$$\mathbf{m}(t) = \int d\theta \hat{\mathbf{r}}(\theta) \int dp f(\theta, p, t). \tag{6}$$

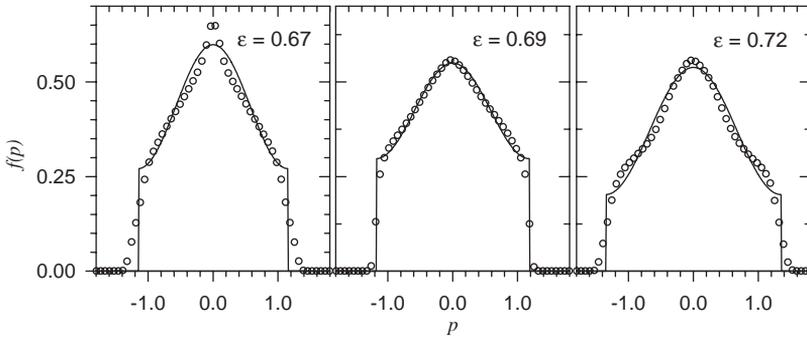


Fig. 2. Momentum distributions for energies indicated in the figure. Histograms (bin size $\simeq 0.07$) were generated averaging on the time interval (500,2000) and 10 different initial conditions (very small random perturbations of the fully magnetized regular water-bag). In all cases $N = 10^4$. See Section 4 for further details. Full lines are water-bag-plus-cosine fittings to the data.

Liouville equation (5) has to be solved self-consistently with the calculation of $\mathbf{m}(t)$, and is thus formally equivalent to the Vlasov–Poisson system found in plasma and gravitational dynamics [13]. The continuum approximation will be progressively better as N grows, especially when the cloud of particles is spread over a finite region in μ -space. This is the case of the distributions we will analyze in this paper (see Fig. 2). Detailed discussions of Vlasov approximation can be found in Refs. [12,14,15].

3. Dynamical stability

Every spatially homogeneous distribution $f(p)$ is a stationary solution of Vlasov equation (5). The reason is that homogeneity leads to $\mathbf{m} = 0$ and thus momenta do not change. The interesting point is whether these homogeneous solutions are stable or not. There is the well-known Landau analysis [13] that concerns *linear* stability. This linear criterion was recently applied to the HMF by Choi and Choi to discuss ensemble equivalence and quasi-stationarity [16]. Besides the linearity restriction, Landau analysis has the disadvantage of requiring the knowledge of detailed analytical properties of $f(p)$. This excludes, in principle, numerically obtained momentum distributions.

A more powerful stability criterion for homogeneous equilibria has been proposed by Yamaguchi et al. [12]. This is a nonlinear criterion specific to the HMF. It states that $f(p)$ is stable if and only if the quantity

$$I = 1 + \frac{1}{2} \int_{-\infty}^{\infty} dp \frac{f'(p)}{p} \quad (7)$$

is positive (it is assumed that f is an even function of p). Note that this condition is equivalent to the zero frequency case of Landau's recipe [5,16].

The authors of Ref. [12] showed that a distribution that is spatially homogeneous and Gaussian in momentum becomes unstable below the transition energy $\varepsilon_{\text{cr}} = \frac{3}{4}$ (see also Refs. [4,16]), in agreement with analytical and numerical results for finite N systems.

They also showed that homogeneous states with zero-mean uniform $f(p)$ (water-bag) are stable above $\varepsilon = \frac{7}{12} = 0.58\dots$ (see also Refs. [5,16]), and verified that the stability of these Vlasov solutions has a counterpart in the associated particle dynamics.

In the same spirit, it is also worth analyzing the stability of the family of q -exponentials

$$f(p) \propto \exp_q(-\alpha p^2) = [1 - \alpha(1 - q)p^2]^{1/(1-q)}, \tag{8}$$

which allow to scan a wide spectrum of probability distributions, from finite-support to power-law-tailed ones, containing as particular cases the Gaussian ($q = 1$) and the water bag ($q = -\infty$). In Eq. (8), the normalization constant has been omitted and the parameter $\alpha > 0$ is related to the second moment $\langle p^2 \rangle$, which only exists for $q < \frac{5}{3}$. In the homogeneous states of the HMF one has $\langle p^2 \rangle = 2\varepsilon - 1$, as can be easily derived from Eq. (2). Then, the stability indicator I as a function of the energy for the q -exponential family reads

$$I = 1 - \frac{3 - q}{2(5 - 3q)(2\varepsilon - 1)}. \tag{9}$$

Stability occurs for energies above ε_q

$$\varepsilon_q(q) = \varepsilon_{cr} + \frac{q - 1}{2(5 - 3q)}. \tag{10}$$

It is easy to verify that one recovers the known stability thresholds for the uniform and Gaussian distributions, i.e., $\varepsilon_q(-\infty) = \frac{7}{12}$ and $\varepsilon_q(1) = \frac{3}{4}$. We remark that Eq. (10) states that only finite-support distributions, corresponding to $q < 1$, are stable below ε_{cr} . This agrees with numerical studies in the QSS regimes of the HMF (see Fig. 2).

The QSS we discuss in this paper are obtained from particular initial conditions: all angles are equal and momenta distributed uniformly with zero mean; energy is chosen slightly below ε_{cr} . After an initial “violent relaxation” (the initial state is highly inhomogeneous), the system evolves to a QSS characterized by spatial homogeneity and momentum distributions with cutoffs [9]. However, the resulting distributions of momenta are not uniform any more. Some structure appears on top of a slightly smoothed water-bag (see Fig. 2). A very simple family of functions exhibiting the basic features of the observed $f(p)$ is the water-bag plus cosine:

$$f(p) = \begin{cases} 1/(2p_{max}) + B \cos(\pi p/p_{max}) & \text{for } p \in [-p_{max}, p_{max}], \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The stable functions $f(p)$ occupy a region in parameter space (p_{max}, B) defined by three boundary curves (see Fig. 3). The left-top boundary is given by $I = 0$ and the other two are associated to positivity, i.e., $f(p) = 0$. The metastable states obtained from numerical simulations of finite systems (to be described below) can be approximately fitted by water-bag-plus-cosine functions. Remarkably, the resulting pairs (p_{max}, B) fall close to the boundary of Vlasov stability, and exit the stability region for energies below the limiting value $\varepsilon \simeq 0.68$.

Fig. 3 also informs that for low energies, $\varepsilon < \frac{7}{12}$, there is a qualitative change in the shape of stable homogeneous distributions: they change the maximum at $p = 0$ by a

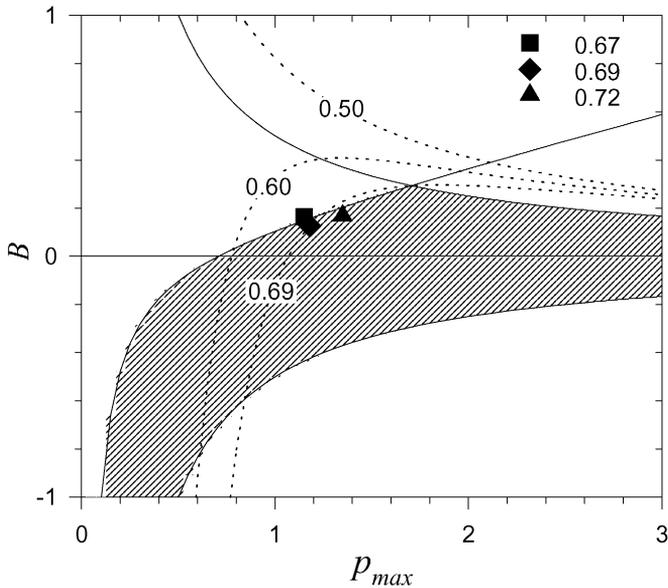


Fig. 3. Stability diagram for water-bag-plus-cosine momentum distributions. The shadowed area corresponds to the stability domain. Also shown are three iso-energetic subfamilies with $\epsilon=0.50, 0.60, 0.69$ (dotted lines). Symbols correspond to quasi-stationary states obtained numerically, for energies $\epsilon = 0.67, 0.69, 0.72$ and $N = 10^4$ (see Section 4 for numerical details).

minimum since $B < 0$. As $\epsilon \rightarrow \frac{1}{2}$, the stable distribution resembles a couple of delta functions approaching the origin.

4. Numerical results

The application of the stability criterion to a discrete distribution requires some coarse-graining. We will consider smoothed distributions

$$f_\gamma(p) = \frac{1}{N} \sum_{i=1}^N \delta_\gamma(p - p_i) \tag{12}$$

with Lorentzian delta functions

$$\delta_\gamma(x) = \frac{\gamma}{\pi} \frac{1}{x^2 + \gamma^2} . \tag{13}$$

The coarse-graining parameter γ corresponds to an intermediate scale, between microscopic and macroscopic (to be specified). We will also impose parity, $f(p) = f(-p)$, so that our smoothed distribution will be calculated as

$$f_\gamma(p) = \frac{\gamma}{\pi N} \sum_{p_i > 0} \left[\frac{1}{(p - p_i)^2 + \gamma^2} + \frac{1}{(p + p_i)^2 + \gamma^2} \right] . \tag{14}$$

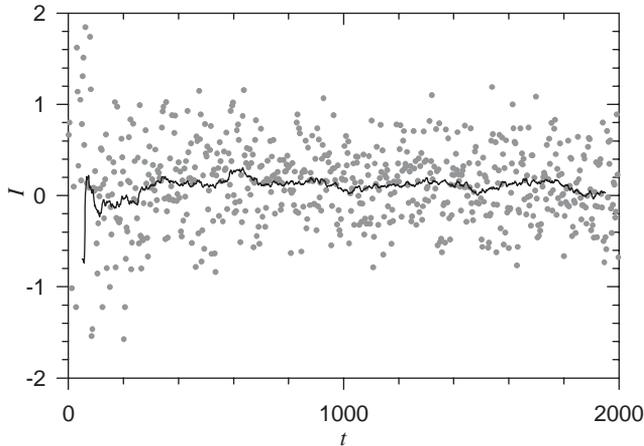


Fig. 4. Example of stability as a function of time, for the fully magnetized regular water-bag initial condition, with $\varepsilon = 0.69$, $N = 10^4$, and smoothing $\gamma = 0.05$. The full line corresponds to a running average over a time window of width $\Delta t = 100$, indicating quasi-stationarity in the interval $500 < t < 2000$.

Accordingly, the stability indicator becomes simply

$$I = 1 + \frac{1}{N} \sum_{p_i > 0} \frac{p_i^2 - \gamma^2}{(p_i^2 + \gamma^2)^2}. \quad (15)$$

All simulations have been done with $N = 10^4$ particles. We have considered water-bag initial conditions. All particles are set at $\theta = 0$, corresponding to maximum magnetization modulus $m = 1$. Momenta were distributed on a regular lattice, symmetrically around $p = 0$. Our experience indicates that this “crystalline” distribution behaves like a larger system, say $N = 10^5$, where the p ’s are extracted randomly from a uniform distribution. Initial states were propagated according to Hamilton equations, approximated by a symplectic fourth-order scheme [17]. We registered the quantities of interest $m(t)$ (homogeneity indicator) and $p_i(t)$ (input to the stability indicator). Strictly speaking, $m = 0$ does not imply that the states are homogeneous. However, the sudden relaxation that leads to the present QSS mixes particles [8] in such a way that $m = 0$ and spatial homogeneity are expected to be synonymous.

Concerning the calculation of smoothed distributions, we chose a coarse-graining parameter $\gamma = 0.05$, after verifying that such a choice erases microscopic fluctuations, but preserves macroscopic features of the QSS momentum distributions (like those shown in Fig. 2). Anyway, some tests were run with $\gamma = 0.025$ and $\gamma = 0.1$ to check that our results are not substantially changed by the precise choice of the smoothing (see inset in Fig. 5). We also eliminated rapid time fluctuations by doing an additional average in the time window $500 < t < 2000$, a typical domain of quasi-stationarity after initial relaxation of the water-bag (see Fig. 4).

The sensitivity to initial conditions (of the microscopic state of the system) after times like those considered here ($t \approx 1000$) is enormous. To be sure that the effect we are observing is statistically robust, we also did an average on initial conditions.

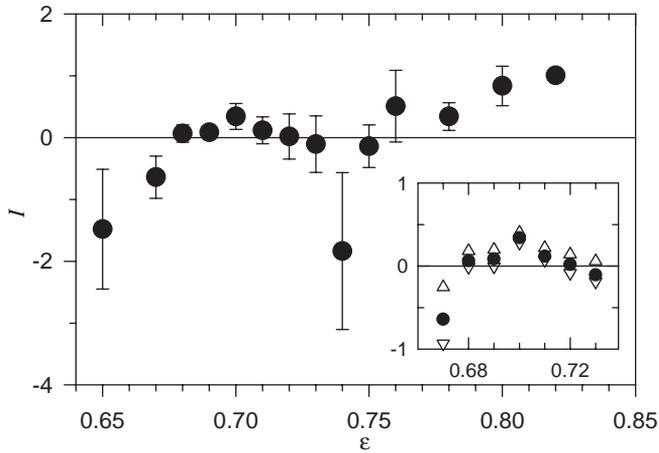


Fig. 5. Stability versus energy for quasi-stationary states. Note that I goes through zero around $\varepsilon \simeq 0.68$. The smoothing parameter is $\gamma = 0.05$ ($N = 10^4$). We averaged over 10 initial conditions and in the time interval $500 < t < 2000$. Inset: I vs. ε for $\gamma = 0.025$ (∇), 0.05 (\bullet) and 0.1 (\triangle).

These were obtained by slightly perturbing the regular water-bag with a small random component: $p'_i = p_i(1 + 10^{-4}\xi)$, where ξ is random uniform in $[-1, 1]$. Our stability analysis of the coarse-grained quasi-stationary states of the HMF is summarized in Fig. 5. There one sees that stability is positive for energies above $\varepsilon \simeq 0.68$, except for the small region $0.73 < \varepsilon < 0.75$. We believe that this is probably a spurious effect due to the proximity of the ferromagnetic phase transition; further analyses, involving longer times, or larger system sizes, are necessary for elucidating this question. The fact that the stability indicator becomes negative below $\varepsilon \simeq 0.68$ signals the disappearance of the homogeneous metastable phase at that energy. Simultaneously with I crossing zero at that point, the magnetization rises from negligible values to finite ones, of the order of those corresponding to equilibrium (Fig. 1). This stability picture is consistent with the analysis of the water-bag-plus-cosine family (Section 3), showing that distributions like those observed in QSS live close to a stability border, and that this border is traversed when energy goes down a limiting value. The symbols in Fig. 3 were obtained by fitting a water-bag-plus-cosine to the numerical histograms of Fig. 2.

A comment about negative stability is in order. The present stability test only applies to homogeneous states. However, below $\varepsilon = 0.68$ the measured distributions are evidently inhomogeneous ($m \neq 0$). In these cases, negative stability refers to hypothetical homogeneous states having the measured $f(p)$.

5. Conclusions

Yamaguchi et al. [12] conjectured that stable homogeneous solutions of Vlasov equation lead, via discretization, to quasi-stationary states of Hamiltonian dynamics, and

showed that it is indeed the case for some families of states of the HMF. We have shown that the reciprocal may be also true: metastable states of the discrete Hamiltonian dynamics lead, after smoothing, to stable solutions of the Vlasov equation (this was verified for the homogeneous QSS).

Moreover, our analysis indicates that the disappearance of the homogenous family (resulting from the violent relaxation of special water-bag initial conditions) at $\varepsilon \simeq 0.68$ can be interpreted as a reflection of the loss of Vlasov stability at that critical energy. This interpretation is supported by similar evidence resulting from the analytical study of the water-bag-plus-cosine family.

Despite providing interesting information, the results we presented are not definitive. Some points remain unclear and deserve further investigation, e.g., the stability properties close to the ferromagnetic transition. Another point, perhaps more important, concerns the quantification of spatial homogeneity. A null-squared magnetization indicates the absence of spatial structures on the largest scale. A more stringent test would require the verification that higher-order Fourier coefficients of the particle density tend to zero in the large N limit.

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