# Nonextensive statistical mechanics and economics

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## Abstract

Ergodicity, this is to say, dynamics whose time averages coincide with ensemble averages, naturally leads to Boltzmann–Gibbs (BG) statistical mechanics, hence to standard thermodynamics. This formalism has been at the basis of an enormous success in describing, among others, the particular stationary state corresponding to thermal equilibrium. There are, however, vast classes of complex systems which accommodate quite badly, or even not at all, within the BG formalism. Such dynamical systems exhibit, in one way or another, nonergodic aspects. In order to be able to theoretically study at least some of these systems, a formalism was proposed 14 years ago, which is sometimes referred to as nonextensive statistical mechanics. We briefly introduce this formalism, its foundations and applications. Furthermore, we provide some bridging to important economical phenomena, such as option pricing, return and volume distributions observed in the financial markets, and the fascinating and ubiquitous concept of risk aversion. One may summarize the whole approach by saying that BG statistical mechanics is based on the entropy $S_{BG} = -k \sum_i p_i \ln p_i$, and typically provides *exponential laws* for describing stationary states and basic time-dependent phenomena, while nonextensive statistical mechanics is instead based on the entropic form $S_q = k(1 - \sum_i p_i^q)/(q - 1)$ (with $S_1 = S_{BG}$), and typically provides, for the same type of description, (asymptotic) *power laws*.
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Connections between dynamics and thermodynamics are far from being completely elucidated. Frequently, statistical mechanics are presented as a self-contained body, which could dispense dynamics from its formulation. This is an unfounded assumption (see, for instance, Ref. [1] and references therein). Questions still remain open even for one of its most well-established equilibrium concepts, namely the Boltzmann–Gibbs (BG) factor $e^{-E_i/kT}$, where $E_i$ is the energy associated with the $i$th microscopic state of a conservative Hamiltonian system, $k$ is Boltzmann constant, and $T$ the absolute temperature. For example, no theorem exists stating the necessary and sufficient conditions for the use of this celebrated and ubiquitous factor to be justified. In the mathematician Takens' words [2]:

> The values of $p_i$ are determined by the following dogma: if the energy of the system in the $i$th state is $E_i$ and if the temperature of the system is $T$ then: $p_i = \exp\{-E_i/kT\}/Z(T)$, where $Z(T) = \sum_i \exp\{-E_i/kT\}$, (this last constant is taken so that $\sum_i p_i = 1$). This choice of $p_i$ is called *Gibbs distribution*. We shall give no justification for this dogma; even a physicist like Ruelle disposes of this question as "deep and incompletely clarified".

One possible reason for this essential point having been poorly emphasized is that when dealing with short-range interacting systems, BG thermodynamical equilibrium may be formulated without much referring to the underlying dynamics of its constituents. One rarely finds in textbooks much more than a quick mention to *ergodicity*. A full analysis of the microscopic dynamical requirements for ergodicity to be ensured is still lacking, in spite of the pioneering studies of Krylov [3] (For full details see Krylov, 1979). In his words:

> In the present investigation, the notion of ergodicity is ignored. I reject the ergodical hypothesis completely: it is both insufficient and unnecessary for statistics. I use, as starting point, the notion of motions of the mixing type, and show that the essential mechanical condition for the applicability of statistics consists in the requirement that in the phase space of the system all the regions with a sufficiently large size should vary in the course of time in such a way that while their volume remains constant—according to Liouville's theorem—their parts should be distributed over the whole phase space (more exactly over the layer, corresponding to given values of the single-valued integrals of the motion) with a steadily increasing degree of uniformity. [...] The main condition of mixing, which ensures the fulfillment of this condition, is a sufficiently rapid divergence of the geodetic lines of this Riemann space (that is, of the paths of the system in the $n$-dimensional configuration space), namely, an exponential divergence (cf. Nopf[1]).

Another possibly concomitant reason no doubt is the enormous success, since more than one century, of BG statistical mechanics for many systems. However, as complex systems become more and more prominent in the front line of research [by *complex* we mean here the presence of at least one, typically more, of the following features: long-range interparticle interactions, long-term microscopic or mesoscopic memory,

(multi)fractal nature of a pertinent subset of phase-space where the system remains long time or forever, small-world or scale-free networking; see Ref. [4] for a comprehensive introduction to the subject], this fundamental issue starts being revised (see Ref. [5] and references therein).

Indeed, a significant amount of systems, e.g., turbulent fluids ([6,7] and references therein), electron–positron annihilation [8], cosmic rays [9], economics [10–12, see also Michael and Johnsson, 2002] motion of *Hydra viridissima* [13], kinetic theory [14], classical chaos [15], quantum chaos [16], quantum entanglement [17], long-range-interacting many-body classical Hamiltonian systems ([18] and references therein), internet dynamics [19], and others, are known nowadays which hardly, or not at all, accommodate within BG statistical mechanical concepts. Systems like these have been successfully handled with the functions and concepts which naturally emerge within nonextensive statistical mechanics [20–22].

The basic building block of nonextensive statistical mechanics is the nonextensive entropy [20]

$$S_q \equiv k \, \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \quad (q \in \mathscr{R}) \,. \tag{1}$$

The entropic index $q$ characterizes the statistics we are dealing with; $q = 1$ recovers the usual BG expression, $S_1 = -k \sum_{i=1}^{W} p_i \ln p_i$. We may think of $q$ as a biasing parameter: $q < 1$ privileges rare events, while $q > 1$ privileges common events. Indeed, $0 < p < 1$ raised to a power $q < 1$ yields a value *larger* than $p$, and the relative increase $p^q/p = p^{q-1}$ is a *decreasing* function of $p$, i.e., values of $p$ closer to 0 (rare events) are benefited. Correspondingly, for $q > 1$, values of $p$ closer to 1 (common events) are privileged. Therefore, the BG theory (i.e., $q = 1$) is the unbiased statistics. A concrete consequence of this is that the BG formalism yields *exponential* equilibrium distributions (and time behavior of typical relaxation functions), whereas nonextensive statistics yields (asymptotic) *power-law* distributions (and relaxation functions). Since the BG exponential is recovered as a limiting case, we are talking of a *generalization*, not an alternative.

The optimization of the entropic form (1) under appropriate constraints [20,21] yields, for the stationary state, the following distribution of probabilities:

$$p_i = \frac{[1 - (1 - q)\beta_q(E_i - U_q)]^{1/(1-q)}}{Z_q} \,, \tag{2}$$

where

$$Z_q \equiv \sum_j [1 - (1 - q)\beta_q(E_j - U_q)]^{1/(1-q)} \tag{3}$$

with

$$U_q \equiv \frac{\sum_j p_j^q E_j}{\sum_j p_j^q} \tag{4}$$

and

$$\beta_q \equiv \frac{\beta}{\sum_j p_j^q} \,, \tag{5}$$

$\beta$ being the optimization Lagrange parameter associated with the generalized internal energy $U_q$. Eq. (2) can be rewritten as

$$p_i \propto [1 - (1 - q)\beta' E_i]^{1/(1-q)} \equiv e_q^{-\beta' E_i} \,, \tag{6}$$

where $\beta'$ is a renormalized inverse "temperature", and the *q-exponential function* is defined as $e_q^x \equiv [1 + (1 - q)x]^{1/(1-q)} = 1/[1 - (q - 1)x]^{1/(q-1)}$ (with $e_1^x = e^x$). This function replaces, in a vast number of relations and phenomena, the usual BG factor. In particular, the ubiquitous Gaussian distribution $\propto e^{-ax^2}$ becomes generalized into the distribution $\propto e_q^{-a_q x^2} = 1/[1 + (q - 1)a_q x^2]^{1/(q-1)}$ (fat-tailed if $q > 1$, and compact support if $q < 1$).

The use of the concepts and methods of statistical mechanics and thermodynamics in economics has long proved to be a fruitful one ([23] and references therein). It has even created a specific field of research, whose name—*Econophysics*—has been proposed by Stanley a few years ago. Let us now briefly review three recent applications of the ideas associated with nonextensive statistical mechanics to phenomena in economics, namely a simple trading model which takes into account risk aversion [10], a generalization of the Black–Scholes equation for pricing options [11], and a phenomenological description of distributions of returns and volumes in the real market [12].

*Application to risk aversion*:

The works by Kahneman, Tversky and others [24] have put into quantitative evidence the generic and enormous importance of risk aversion (when one expects to win) and risk seeking (when one expects to lose) in trading. The biased averages which naturally emerge in nonextensive statistical mechanics resemble those proposed in *prospect theory*, and constitute a tool that can be used to make [10] simple, and relatively realistic, models for a stock exchange or similar forms of trading. In fact, in Ref. [10], an automaton is introduced which simulates monetary transactions among operators with different attitudes under risky choices. Each operator is characterized by a parameter $q$ which measures his(her) attitude under risk; this index $q$ is the entropic one which plays a central role in nonextensive statistical mechanics. Elementary operations are of the standard type used in hypothetical choice problems that exhibit risk aversion [24]. By following the asset position of the operators, it is possible to conclude on the consequences of each particular attitude on the *dynamics* of economic operations.

*Application to option pricing*:

The celebrated Black–Scholes equation provides, in an explicit manner, the prices of options. Its basic formulation assumes the presence of a Gaussian noise, which simplifies the mathematical treatment on one hand, but yields results which diverge from those observed in real markets. One of us [11] has recently generalized this equation by assuming non-Gaussian fluctuations, evolving anomalously in time according to a nonlinear Fokker–Planck equation. Such a noise process can be modeled as the result of a standard Gaussian process with statistical feedback, of the type discussed in Ref. [25]. The degree of feedback is characterized by the nonextensive entropic index $q$; if $q > 1$, then rare events will lead to large fluctuations, whereas more common events will result in more moderate fluctuations. This model of stock returns is consistent with
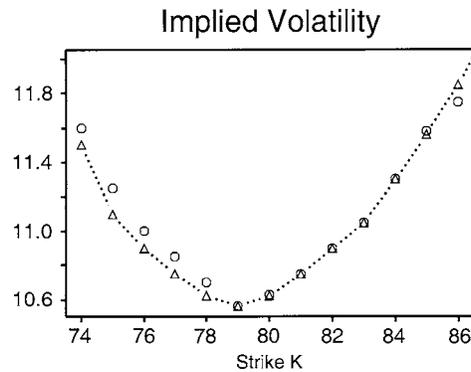
## Implied Volatility



Fig. 1. Implied volatilities as a function of the strike price for call options on JY currency futures, traded on May 16, 2002, with 147 days left to expiration. In this example, the current price of a contract on Japanese futures is $79, and the risk-free rate of return is 5.5%. Circles correspond to volatilities implied by the market, whereas triangles correspond to volatilities implied by our model with $q = 1.4$ and $\sigma = 10.2\%$. The dotted line is a guide to the eye.

empirical observations of the distribution of returns. It is used as a model of stock price fluctuations, from which the fair value of options on the underlying stock can then be derived [11]. The results are appreciably more realistic than the usual Black–Scholes equation which is recovered as the $q = 1$ particular case.

This extended model, which preserves the practical advantage of having explicit closed-form solutions, has proved to be a quite satisfactory one. The theoretical option prices predicted by this model, using $q = 1.4$ which models well the underlying returns distribution, are in very good agreement with empirically observed option prices. Let us look at the example of European call options. Such options are the right to buy, at expiration time $T$, the underlying stock at an agreed upon price $K$, called the strike price. Whereas the standard Black–Scholes equation must use a different value of the volatility $\sigma$ for each value of the option strike price in order to produce theoretical values which match empirical ones, the $q = 1.4$ model uses *just one value of $\sigma$ across all strikes*. For the standard Black–Scholes model, a plot of $\sigma$ versus the strike $K$ forms a convex curve known as the volatility smile. We can use the $q = 1.4$ model with a fixed $\sigma$ to produce theoretical option prices, and we can then find those $\sigma$ which the standard Black–Scholes model must use in order to produce option prices which match the $q = 1.4$ ones. A comparison of such a theoretically obtained volatility smile with the one observed from market data will reflect how closely the $q = 1.4$ model fits real option prices. As can be seen in Fig. 1, where the option smile for Japanese Yen futures is studied, there is a very close agreement.

*Application to financial returns and volumes*:

In the last few years, the easier availability of databases with detailed transaction histories has enabled an increasing interest in the empirical study of distributions of high-frequency financial variables (see, e.g. [26–30]). We are concerned in this section with two types of single-stock variables defined over fixed time intervals: *returns*,
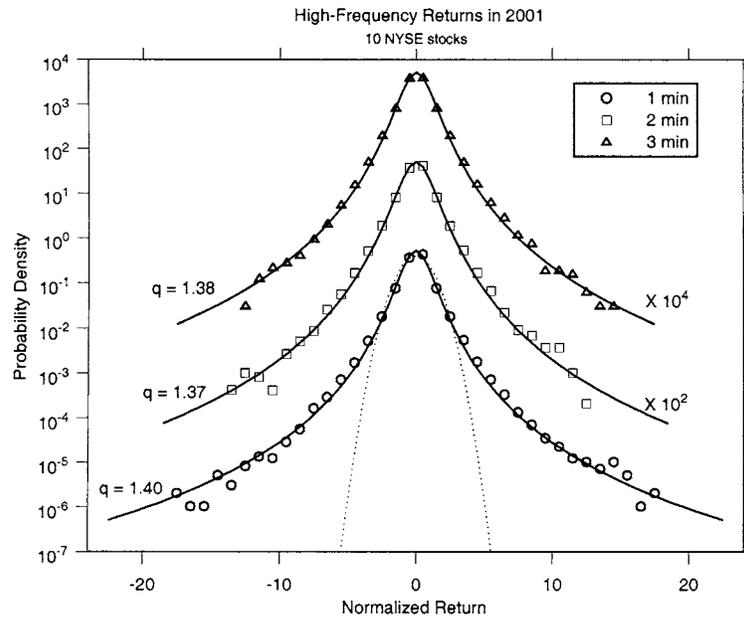
Fig. 2. Empirical distributions (points) and $q$-Gaussians (solid lines) for normalized returns of the 10 top-volume stocks in the NYSE in 2001. The dotted line is the Gaussian distribution. The 2- and 3-min curves are moved vertically for display purposes.

defined here as logarithmic relative price changes, and *volumes*, i.e., numbers of shares traded.

When *normalized returns* (after, for each stock, subtracting their mean and dividing by their standard deviation) are measured over intervals of a few minutes, their distributions are very well fitted by $q$-Gaussians with $q \simeq 1.4$. For long periods (months or years), it seems that values of $q$ that approach the Gaussian limit ($q = 1$) provide an adequate description. Results for high-frequency normalized returns are presented in Figs. 2 and 3 [12] for the 10 top-volume stocks in 2001 in each of the two largest US exchanges: the New York Stock Exchange (NYSE) and the National Association of Security Dealers Automatic Quotation (NASDAQ).

Similar results are obtained for the corresponding distributions of *normalized volumes* (divided by their means), by fitting them with $q$-exponentials multiplied by a simple power of the normalized volume. Typical examples are presented in Figs. 4 and 5 [12]. Previous studies have shown that volumes present a power-law behavior at high values [29], while *volatilities* present both a power-law behavior at high values [26] and a log-normal behavior at smaller values [26,30]. Our curves are reminiscent of the behavior of volatility curves, but we suggest, in addition, a new power-law regime at low volumes and propose a functional form that unifies the description of the three regimes. A model whose solution would be this type of distribution would of course be very welcome.
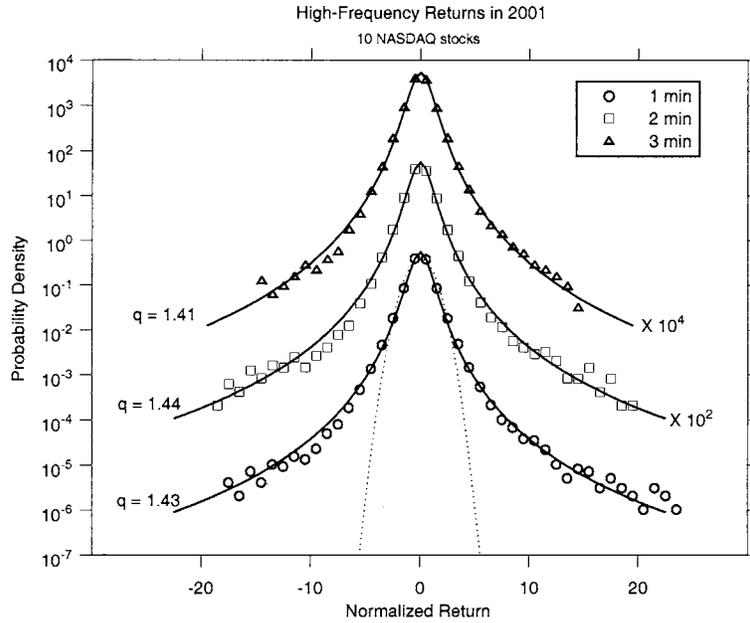
High-Frequency Returns in 2001

10 NASDAQ stocks



Fig. 3. Same as Fig. 2, but for the NASDAQ.

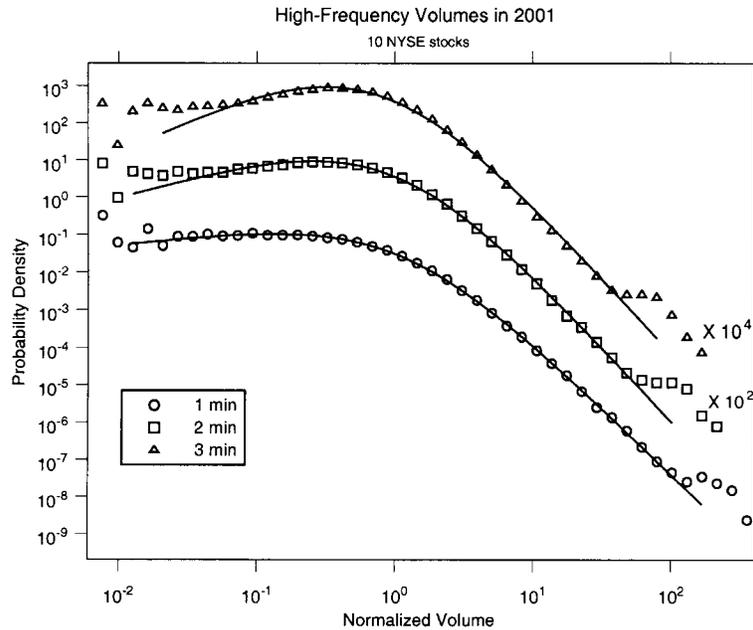High-Frequency Volumes in 2001

10 NYSE stocks



Fig. 4. Empirical distributions (points) and $q$-exponential-like fits (solid lines) for normalized volumes of the 10 top-volume stocks in the NYSE in 2001.

High-Frequency Volumes in 2001
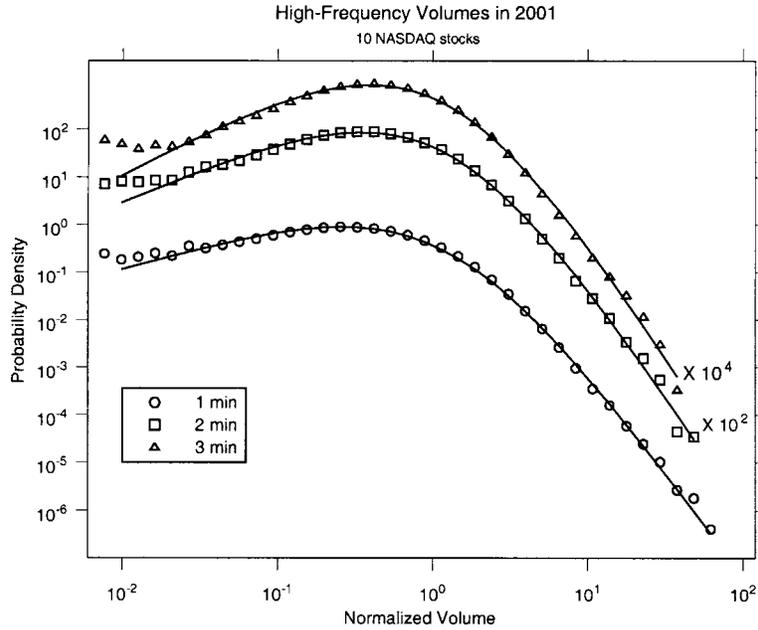
10 NASDAQ stocks



Fig. 5. Same as Fig. 4, but for the NASDAQ.

With regard to the specific distributions associated with returns (Figs. 2 and 3), and similar ones ubiquitously observed in nature, a relevant comment is appropriate. The two basic versions of the central limit theorem (CLT) are the following: (1) A convoluted distribution with a finite second moment approaches, in the limit of $N \to \infty$ convolutions, a Gaussian attractor. (2) A convoluted distribution with a divergent second moment approaches, in the same limit, a Lévy distribution $L_\gamma(x)$ (with $2 > \gamma > 0$). If we consider, for example, the $q$-Gaussian distributions indicated above, then (a) if $q < \frac{5}{3}$, the attractor is a Gaussian (i.e., $\gamma = 2$), (b) if $\frac{5}{3} < q < 3$, the attractor is a Lévy distribution, satisfying $q = (3 + \gamma)/(1 + \gamma)$, and (c) if $q = \frac{5}{3}$ we have a marginal case involving logarithmic corrections. The index $q$ cannot exceed 3 if the norm is to remain finite.

Can we then conclude that *all* stationary distributions found in nature should be of either the Gaussian or the Lévy type? By no means! It would be so only if the unique dynamics emerging in nature were the convolution dynamics, i.e., with no memory across successive steps. In other words, it would be so if the Laplacian term of the associated Fokker–Planck equation in free space were either a standard second derivative or a fractional derivative. In the first case, the solution is a Gaussian; in the second case (for a derivative of order $\gamma$), it is a Lévy distribution. (We assume throughout this discussion the standard first time derivative.)

Actually, much more complex and rich dynamics clearly exist in nature, for example, those associated with a variety of nonlinear Fokker–Planck equations involving
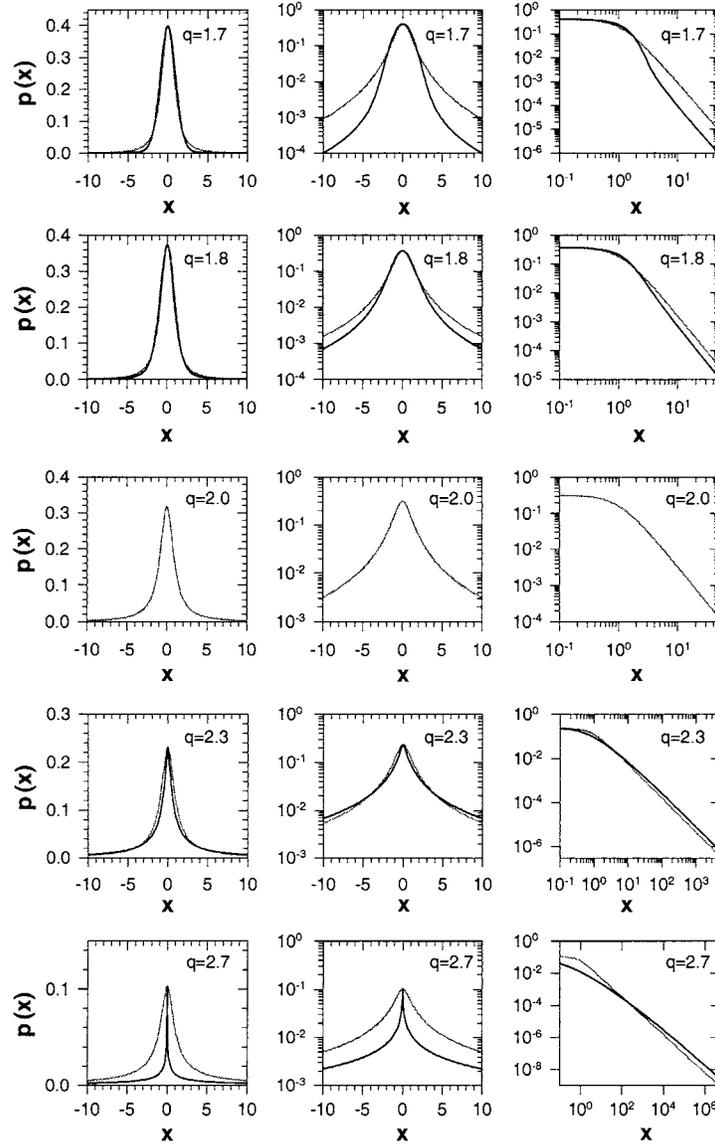
Fig. 6. Lévy distributions $L_\gamma(x) \equiv 1/2\pi \int_{-\infty}^{\infty} dk \cos kx\, e^{-\alpha|k|^\gamma}$, with $0 < \gamma < 2$ and $\alpha > 0$ (black curves), and $q$-Gaussians $P_q(x) \equiv [1 - (1-q)\beta x^2]^{1/(1-q)}/Z_q$, with $\frac{5}{3} < q < 3$, $\beta > 0$ and $Z_q = \sqrt{\pi/(\beta(q-1))}\Gamma((3-q)/2(q-1))/\Gamma(1/(q-1))$ (gray curves). Parameters $(q, \gamma)$ are related through $q = (\gamma+3)/(\gamma+1)$ so that the tails of both distributions decay with the same power-law exponent. Without loss of generality, we have taken $\beta = 1$ which corresponds to a simple rescaling; $\alpha$ was chosen such that $P_q(0) = L_\gamma(0)$.

nontrivial correlations, multiplicative noise and other effects [31,32], whose exact solutions are $q$-Gaussians.

This remark might seem trivial (it has in fact been already mentioned in Ref. [32]), but this point has been overseen by some authors (e.g., Ref. [33]). To clarify this, recall that the simple convolution CLT (in both the traditional and the Lévy–Gnedenko versions) only allows, in the large $N$ limit, for $q = 1$ (Gaussians) and $q > \frac{5}{3}$ (Lévy distributions). It does not allow in any way for fat-tailed distributions associated with $1 < q < \frac{5}{3}$. But it happens that, in many cases, complex systems in both nature and the social sciences seem to have a strong "inclination" precisely for that interval! Such is the case for fully developed turbulence in Couette–Taylor experiments [6] (with $1 \leqslant q \leqslant 1.2$ for Newtonian turbulence and $q \simeq 1.5$ for Lagrangian turbulence), *Hydra viridissima* [13] ($q \simeq 1.5$), electron–positron annihilation [8] ($1 \leqslant q \lesssim 1.3$), cosmic rays [9] ($q \simeq 1.2$), and finance [12] ($q \simeq 1.4$–1.5), among others. There are, however, other situations in nature for which the case is not so clear-cut, although still distinguishable by, e.g., a careful analysis of the behavior near the origin. See Fig. 6 for comparison.

In the specific case of high-frequency financial observations, other studies [27,28] support the facts that (a) variances are finite and (b) the exponent in the power-law tails lies outside the stable Lévy interval $2 > \gamma > 0$. These observations, together with the obvious presence of long tails, clearly demonstrate the inadequacy of CLT distributions (Lévy or Gaussian) to describe high-frequency returns. In contrast, the use of $q$-Gaussians with $1 < q < \frac{5}{3}$ is consistent both with finite variances and with the presence of temporal autocorrelations (or memory effects) in the dynamics of financial systems.

## References

[1] E.G.D. Cohen, Physica A 305 (2002) 19.
[2] F. Takens, in: H.W. Broer, F. Dumortier, S.J. van Strien, F. Takens (Eds.), Structures in Dynamics—Finite Dimensional Deterministic Studies, North-Holland, Amsterdam, 1991, p. 253.
[3] N. Krylov, Nature 153 (1944) 709;
    N.S. Krylov, in: A.B. Migdal, Ya.G. Sinai, Yu.L. Zeeman (Eds.), Works on the Foundations of Statistical Physics (trans.), Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1979.
[4] M. Gell-Mann, The Quark and the Jaguar: Adventures in the Simple and the Complex, W.H. Freeman, New York, 1999.
[5] M. Baranger, Physica A 305 (2002) 27.
[6] C. Beck, G.S. Lewis, H.L. Swinney, Phys. Rev. E 63 (2001) 035 303;
    C. Beck, Phys. Rev. Lett. 87 (2001) 180 601.
[7] T. Arimitsu, N. Arimitsu, Physica A 305 (2002) 218.
[8] I. Bediaga, E.M.F. Curado, J. Miranda, Physica A 286 (2000) 156.
[9] C. Tsallis, J.C. Anjos, E.P. Borges, Fluxes of cosmic rays: a delicately balanced anomalous stationary state, astro-ph/0203258, 2002.

[10] C. Anteneodo, C. Tsallis, A.S. Martinez, Europhys. Lett. 59 (2002) 635.

[11] L. Borland, Phys. Rev. Lett. 89 (2002) 098 701;
     L. Borland, Quant. Finance 2 (2002) 415.

[12] R. Osorio, L. Borland, C. Tsallis, in: M. Gell-Mann, C. Tsallis (Eds.), Nonextensive Entropy—
     Interdisciplinary Applications, Oxford University Press, Oxford, 2003, in press;
     F. Michael, M.D. Johnson, Financial Market Dynamics, Physica A 320 (2003) 525.

[13] A. Upadhyaya, J.-P. Rieu, J.A. Glazier, Y. Sawada, Physica A 293 (2001) 549.

[14] J.A.S. de Lima, R. Silva, A.R. Plastino, Phys. Rev. Lett. 86 (2001) 2938.

[15] C. Tsallis, A.R. Plastino, W.-M. Zheng, Chaos, Solitons Fract. 8 (1997) 885;
     U.M.S. Costa, M.L. Lyra, A.R. Plastino, C. Tsallis, Phys. Rev. E 56 (1997) 245;
     M.L. Lyra, C. Tsallis, Phys. Rev. Lett. 80 (1998) 53;
     U. Tirnakli, C. Tsallis, M.L. Lyra, Eur. Phys. J. B 11 (1999) 309;
     V. Latora, M. Baranger, A. Rapisarda, C. Tsallis, Phys. Lett. A 273 (2000) 97;
     F.A.B.F. de Moura, U. Tirnakli, M.L. Lyra, Phys. Rev. E 62 (2000) 6361;
     U. Tirnakli, G.F.J. Ananos, C. Tsallis, Phys. Lett. A 289 (2001) 51;
     H.P. de Oliveira, I.D. Soares, E.V. Tonini, Physica A 295 (2001) 348;
     F. Baldovin, A. Robledo, Europhys. Lett. 60 (2002) 518;
     F. Baldovin, A. Robledo, Phys. Rev. E 66 (2002) 045104(R);
     E.P. Borges, C. Tsallis, G.F.J. Ananos, P.M.C. Oliveira, Phys. Rev. Lett. 89 (2002) 254 103;
     U. Tirnakli, Physica A 305 (2002) 119;
     U. Tirnakli, Phys. Rev. E 66 (2002) 066 212.

[16] Y. Weinstein, S. Lloyd, C. Tsallis, Phys. Rev. Lett. 89 (2002) 214 101.

[17] S. Abe, A.K. Rajagopal, Physica A 289 (2001) 157;
     C. Tsallis, S. Lloyd, M. Baranger, Phys. Rev. A 63 (2001) 042 104;
     C. Tsallis, P.W. Lamberti, D. Prato, Physica A 295 (2001) 158;
     F.C. Alcaraz, C. Tsallis, Phys. Lett. A 301 (2002) 105;
     C. Tsallis, D. Prato, C. Anteneodo, Eur. Phys. J. B 29 (2002) 605;
     J. Batle, A.R. Plastino, M. Casas, A. Plastino, Conditional $q$-entropies and quantum separability:
     a numerical exploration, quant-ph/0207129, 2002.

[18] C. Anteneodo, C. Tsallis, Phys. Rev. Lett 80 (1998) 5313;
     V. Latora, A. Rapisarda, C. Tsallis, Phys. Rev. E 64 (2001) 056 134;
     A. Campa, A. Giansanti, D. Moroni, in: G. Kaniadakis, M. Lissia, A. Rapisarda (Eds.), Non Extensive
     Statistical Mechanics and Physical Applications, Physica A 305 (2002) 137;
     B.J.C. Cabral, C. Tsallis, Phys. Rev. E 66 (2002) 065101(R);
     E.P. Borges, C. Tsallis, in: G. Kaniadakis, M. Lissia, A. Rapisarda (Eds.), Non Extensive Statistical
     Mechanics and Physical Applications, Physica A 305 (2002) 148;
     A. Campa, A. Giansanti, D. Moroni, C. Tsallis, Phys. Lett. A 286 (2001) 251;
     M.-C. Firpo, S. Ruffo, J. Phys. A 34 (2001) L511;
     C. Anteneodo, R.O. Vallejos, Phys. Rev. E 65 (2002) 016 210;
     R.O. Vallejos, C. Anteneodo, Phys. Rev. E 66 (2002) 021 110;
     M.A. Montemurro, F. Tamarit, C. Anteneodo, Aging in an infinite-range Hamiltonian system of coupled
     rotators, Phys. Rev. E (2003), in press.

[19] S. Abe, N. Suzuki, Itineration of the Internet over nonequilibrium stationary states in Tsallis statistics,
     Phys. Rev. E (2002), in press.

[20] C. Tsallis, J. Stat. Phys. 52 (1988) 479.

[21] E.M.F. Curado, C. Tsallis, J. Phys. A 24 (1991) L69 [Corrigenda 24 (1991) 3187 and 25 (1992) 1019];
     C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534.

[22] S.R.A. Salinas, C. Tsallis (Eds.), Nonextensive statistical mechanics and thermodynamics, Braz. J. Phys.
     29(1) (1999);
     S. Abe, Y. Okamoto (Eds.), Nonextensive Statistical Mechanics and its Applications, Series Lecture
     Notes in Physics, Springer, Berlin, 2001;
     G. Kaniadakis, M. Lissia, A. Rapisarda (Eds.), Non extensive statistical mechanics and physical
     applications, Physica A (Elsevier, Amsterdam) 305(1/2) (2002);

M. Gell-Mann, C. Tsallis (Eds.), Nonextensive Entropy—Interdisciplinary Applications, Oxford University Press, Oxford, 2003, in press;

H.L. Swinney, C. Tsallis (Eds.), Anomalous Distributions, Nonlinear Dynamics, and Nonextensivity, Physica D (2003), in preparation (An updated bibliography can be found at the web site http://tsallis.cat.cbpf.br/biblio.htm)

[23] R.N. Mantegna, H.E. Stanley, Nature 376 (1995) 46;

R.N. Mantegna, H.E. Stanley, Introduction to Econophysics: Correlations & Complexity in Finance, Cambridge University Press, Cambridge, MA, 2000;

H.E. Stanley, L.A.N. Amaral, S.V. Buldyrev, P. Gopikrishnan, V. Plerou, M.A. Salinger, in: Proceedings of the National Academy of Science, Arthur M. Sackler Colloquium, Self-Organized Complexity in the Physical, Biological, and Social Sciences (23–24 March 2001, Beckman Center, Irvine, CA), Proc. Natl. Acad. Sci. 99 (Suppl. 1) (2002) 2561.

[24] D. Kahneman, A. Tversky, Econometrica 47 (1979) 263;

A. Tversky, D. Kahneman, J. Risk Uncertainty 5 (1992) 297;

A. Tversky, P. Wakker, Econometrica 63 (1995) 1255;

A. Tversky, C.R. Fox, Psychological Review 102 (1995) 269;

R. Gonzalez, G. Wu, Cognitive Psychol. 38 (1999) 129.

[25] L. Borland, Phys. Rev. E 57 (1998) 6634.

[26] Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C.K. Peng, H.E. Stanley, Phys. Rev. E 60 (1999) 1390.

[27] V. Plerou, P. Gopikrishnan, L.A.N. Amaral, M. Meyer, H.E. Stanley, Phys. Rev. E 60 (1999) 6519.

[28] P. Gopikrishnan, V. Plerou, Y. Liu, L.A.N. Amaral, X. Gabaix, H.E. Stanley, Physica A 287 (2000) 362.

[29] P. Gopikrishnan, V. Plerou, X. Gabaix, H.E. Stanley, Phys. Rev. E 62 (2000) R4493.

[30] T.G. Andersen, T. Bollersev, F.X. Diebold, H. Ebens, J. Financial Econom. 63 (2001) 43.

[31] A.R. Plastino, A. Plastino, Physica A 222 (1995) 347;

C. Tsallis, D.J. Bukman, Phys. Rev. E 54 (1996) R2197;

C. Giordano, A.R. Plastino, M. Casas, A. Plastino, Eur. Phys. J. B 22 (2001) 361;

A. Compte, D. Jou, J. Phys. A 29 (1996) 4321;

A.R. Plastino, M. Casas, A. Plastino, Physica A 280 (2000) 289;

C. Tsallis, E.K. Lenzi, in: R. Hilfer et al. (Eds.), Strange Kinetics, Chem. Phys. 284 (2002) 341 [Erratum 287 (2003) 295];

E.K. Lenzi, L.C. Malacarne, R.S. Mendes, I.T. Pedron, Anomalous diffusion, nonlinear fractional Fokker–Planck equation and solutions, Physica A 319 (2003) 245 [cond-mat/0208332];

E.K. Lenzi, C. Anteneodo, L. Borland, Phys. Rev. E 63 (2001) 051 109;

E.M.F. Curado, F.D. Nobre, Derivation of nonlinear Fokker–Planck equations by means of approximations to the master equation, Phys. Rev. E 67 (2003) 021107;

C. Anteneodo, C. Tsallis, Multiplicative noise: a mechanism leading to nonextensive statistical mechanics, cond-mat/0205314, 2002.

[32] M. Bologna, C. Tsallis, P. Grigolini, Phys. Rev. E 62 (2000) 2213.

[33] D.H. Zanette, M.A. Montemurro, Thermal measurements of stationary nonequilibrium systems: a test for generalized thermostatistics, cond-mat/0212327, 2002.