INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF OPTICS B: QUANTUM AND SEMICLASSICAL OPTICS

J. Opt. B: Quantum Semiclass. Opt. **5** (2003) 73–76 PII: S1464-4266(03)53767-3

Enhancement of thermal entanglement in two-qubit XY models

Celia Anteneodo[1] and André M C Souza[1,2]

[1] Centro Brasileiro de Pesquisas Físicas, R Dr Xavier Sigaud 150, 22290-180 RJ, Rio de Janeiro, Brazil

[2] Departamento de Física, Universidade Federal de Sergipe, 49100-000 SE, São Cristóvão, Brazil

E-mail: celia@cbpf.br and amcsouza@ufs.br

Received 12 September 2002, in final form 2 December 2002
Published 21 January 2003
Online at stacks.iop.org/JOptB/5/73

Abstract

We analyse conditions leading to enhancement of thermal entanglement in two-qubit XY models. The effect of including cross-product terms, besides the standard XY exchange interactions, in the presence of an external magnetic field, is investigated. We show that entanglement can be yielded at elevated temperatures by tuning the orientation of the external magnetic field. The details of the intrinsic exchange interactions determine the optimal orientation.

Keywords: Quantum entanglement, thermal states, XY model

1. Introduction

Entanglement implies strong correlations between spatially separated quantum systems that cannot occur classically. By virtue of its nonclassical nature, entanglement has started to be seen in recent years as a phenomenon at the core of future technology, namely, as a potentially useful resource for quantum information processing. Nevertheless, beyond this exciting possibility, quantum entanglement deserves further investigation as it is not a rare phenomenon but it is generic for the states of interacting many-particle systems and, therefore, ubiquitous.

In the preparation of entangled states, the external control performed through changes in temperature, electromagnetic fields etc should not introduce significant levels of decoherence and, at the same time, the required values of the external control parameters must fall within realistic working ranges. Due to the usually large decoherence times, the spin is a property that has been exploited in many implementations requiring entangled states [1]. As a consequence, most of the solid state proposals lie upon two-particle exchange interactions of the Heisenberg type [2], so that the intrinsic two-body Hamiltonian operator is of the form

$$H_o' = \frac{1}{2} \sum_{i=x,y,z} J^i \sigma_i^{(1)} \otimes \sigma_i^{(2)} \quad (1)$$

where $\sigma_i^{(n)}$ are the Pauli matrices of qubit n and J^i the coupling coefficients ($i = x, y, z$). For instance, one finds the isotropic Heisenberg model ($J^x = J^y = J^z$) in spin-coupled quantum dots [3] and donor-atom nuclear spins [4]. Its XY version ($J^x, J^y \neq J^z = 0$) is found in quantum dots in cavities [5], excitons in coupled quantum dots [6], atoms in cavities [7] and quantum Hall systems [8]. Ising-like systems ($J^x \neq J^y = J^z = 0$) have also been considered [9].

Entanglement of two-qubit XY systems, in the presence of a constant external magnetic field $B_z\hat{z}$, has been investigated recently [10, 11]. It was shown that the thermal entanglement of anisotropic samples can be manipulated through changes in B_z. It is our purpose here to analyse the effect that the inclusion of cross-product terms of the form $\sigma_i^{(1)} \otimes \sigma_j^{(2)}$, with $i \neq j$, yields upon thermal entanglement. These terms model weak ferromagnetism and originate from the spin–orbit coupling [2, 12]. Also in the scenario of fermion quantum circuits, the anisotropy in the exchange interaction may be accompanied by cross-product terms [13]. For external control, we will introduce a constant external magnetic field $B = (B_x, B_y, B_z)$.

The degree of entanglement will be gauged through the *concurrence* [14], a measure that, for two-qubit systems, gives the exact frontier between separable and entangled domains. Briefly, the concurrence C is calculated as

$$C = \max\left\{2\max\{\lambda_i\} - \sum_{i=1}^{4} \lambda_i, 0\right\}, \quad (2)$$

with $\{\lambda_i\}$ the square roots of the eigenvalues of the matrix $R = \rho S \rho^* S$, where ρ is the density matrix, $S = \sigma_y^{(1)} \otimes \sigma_y^{(2)}$ and '$*$' stands for complex conjugate. A nonzero concurrence means that the two qubits are entangled, in particular unitary concurrence corresponds to maximally entangled states.

The paper is organized as follows. In section 2 we describe the system. The results are presented and discussed in section 3. Finally, section 4 contains the concluding remarks.

2. The system

In order to study the effect of extra cross-product terms in the XY model, let us consider the following internal Hamiltonian:

$$H_o = \tfrac{1}{2} \sum_{i,j=x,y} a^{ij} \sigma_i^{(1)} \otimes \sigma_j^{(2)}, \tag{3}$$

with $a^{xx} = (1+\gamma)J$ and $a^{yy} = (1-\gamma)J$, where $J \in \Re$ is the coupling constant and $\gamma \in [-1, 1]$ is an anisotropic parameter. Also, let us define the cross-couplings as $a^{xy} = (1+\gamma')K$ and $a^{yx} = (1-\gamma')K$ with $K \in \Re$ the cross-coupling constant and $\gamma' \in [-1, 1]$. Then, H_o, written in terms of the raising and lowering operators $\sigma_\pm^{(n)} = (\sigma_x^{(n)} \pm \mathrm{i}\sigma_y^{(n)})/2$, reads

$$\begin{aligned} H_o = (\gamma J - \mathrm{i}K)\sigma_+^{(1)} \otimes \sigma_+^{(2)} + (J + \mathrm{i}\gamma' K)\sigma_+^{(1)} \otimes \sigma_-^{(2)} \\ + (J - \mathrm{i}\gamma' K)\sigma_-^{(1)} \otimes \sigma_+^{(2)} + (\gamma J + \mathrm{i}K)\sigma_-^{(1)} \otimes \sigma_-^{(2)}. \end{aligned} \tag{4}$$

It is clear now that the inclusion of cross-product terms is equivalent to setting complex coupling constants.

In the presence of the magnetic field B, the complete dimensionless Hamiltonian becomes

$$H = H_o + \tfrac{1}{2} \sum_{i=x,y,z} B_i (\sigma_i^{(1)} \oplus \sigma_i^{(2)}). \tag{5}$$

We consider states of equilibrium at temperature T. Thus, the density matrix is given by $\rho = \exp(-H/T)/Z$ with $Z = \mathrm{Tr}(\exp(-H/T))$ where we have already set the Boltzmann constant $k_B = 1$. The entanglement at these states is called *thermal entanglement* [15].

3. Results

Let us analyse first the case $B_\perp = B_z \neq 0$, $B_\parallel = 0$, where $B_\parallel$ is the component of the field in the xy-plane. The eigenstates of the density matrix ρ are

$$|\psi_\mu^\pm\rangle = [(J + \mathrm{i}\gamma' K)|\uparrow\downarrow\rangle \pm \mu|\downarrow\uparrow\rangle]/N_\mu \tag{6}$$

(with eigenvalues $\pm\mu$, respectively, where $\mu = \sqrt{J^2 + (\gamma' K)^2}$) and

$$|\phi_\nu^\pm\rangle = [(\gamma J - \mathrm{i}K)|\uparrow\uparrow\rangle + (\pm\lambda - B_\perp)|\downarrow\downarrow\rangle]/N_\lambda \tag{7}$$

(with eigenvalues $\pm\lambda$, where $\lambda = \sqrt{\nu^2 + B_\perp^2}$ and $\nu^2 = (\gamma J)^2 + K^2$), where N_μ, N_λ are normalization constants.

The eigenvalues of matrix R are given by

$$\begin{aligned} \lambda_{1,2}^2 &= \exp(\pm 2\mu/T)/Z^2 \\ \lambda_{3,4}^2 &= (1 + 2x^2 \pm 2x\sqrt{1+x^2})/Z^2 \end{aligned} \tag{8}$$

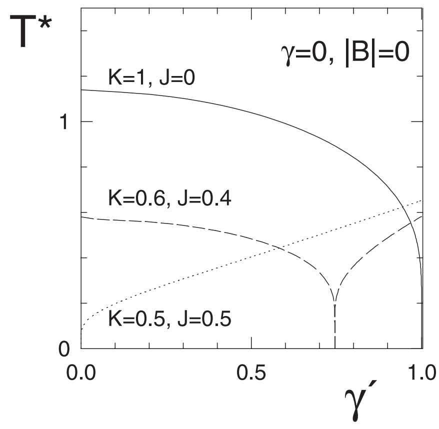

Figure 1. Temperature T^* above which the concurrence vanishes as a function of γ', in the absence of magnetic field $|B| = 0$, for $\gamma = 0$ and different values of (J, K) indicated in the figure.

with $x = (\nu/\lambda)\sinh(\lambda/T)$ and $Z = 2\cosh(\nu/T) + 2\cosh(\lambda/T)$. Then, the concurrence C is straightforwardly computed through equation (2). At $T \to \infty$ or $B_\perp \to \infty$, the entanglement vanishes ($C \to 0$). At $T = 0$, the concurrence is

$$C(T=0) = \begin{cases} 1 & \text{for } \mu > \lambda, \\ (1 - \nu/\lambda)/2 & \text{for } \mu = \lambda, \\ \nu/\lambda & \text{for } \mu < \lambda. \end{cases} \tag{9}$$

Hence, for $T = 0$, there is a jump in the concurrence at $\mu = \lambda$ similarly as already observed in the literature for the case $K = 0, \gamma \neq 0$ [11]. For $\mu > \lambda$, the ground state is $|\psi_\mu^-\rangle$, which is fully entangled. For $\mu < \lambda$, the ground state, $|\phi_\nu^-\rangle$, is partially entangled unless $B_\perp = 0$ in which case it becomes fully entangled. But, at $\mu = \lambda$, the ground state disentangles partially (totally) for $B_\perp \neq 0$ (=0).

The results are not dependent on the sign of the coupling parameters; therefore, in particular, they are valid both for the antiferromagnetic and ferromagnetic cases. Also they are independent of the sign of $B_\perp$.

The picture in the absence of magnetic field ($|B| = 0$) is recovered by setting $B_\perp = 0$ in expressions (6)–(9). In figure 1 we show T^*, the temperature at which the concurrence vanishes, as a function of parameters γ', for different sets of (J, K), in the absence of any external magnetic field. The concurrence is null above T^*. If $\gamma'^2 > 1 - (J/K)^2$, then the threshold temperature T^* grows with increasing γ'. For $|B| = 0$, by interchanging the values of parameters (K, γ') with those of (J, γ), one obtains the same results. Taking this into account, notice that, even for the Ising case ($|\gamma| = 1$), there can be entanglement at finite temperature in the absence of magnetic field ($|B| = 0$), as soon as K is large enough.

Figure 2 exhibits the effect of the external field $B_\perp$ on an isotropic sample ($\gamma = 0$). By increasing $|B_\perp|$, it is possible to increase the threshold temperature T^* provided $K \neq 0$. In [11] it was shown that anisotropic exchange interactions are required for allowing control of T^* by varying the intensity of $B_\perp$. However, the presence of cross-terms enables control of the threshold temperature by varying the intensity of the magnetic field in the z-direction, even in the absence of anisotropy.

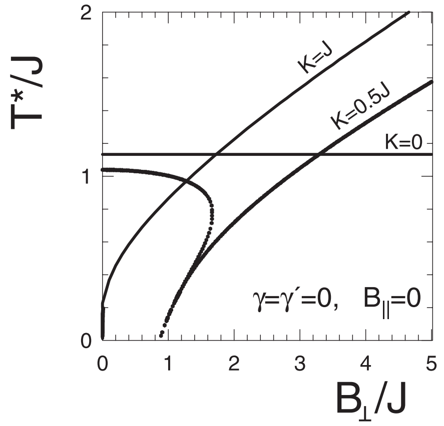

Figure 2. Temperature T^* as a function of $B_\perp$, for the parameters indicated in the figure. The entangled domain is the low-temperature region bounded by the curve and abscissa axis.

If $B_\parallel = 0$ the subspaces spanned by $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ and by $\{|\uparrow\downarrow\rangle, |\uparrow\downarrow\rangle\}$ remain uncoupled, while for $B_\parallel \neq 0$ they mix. Numerical calculations are performed in this case. As expected, the results are only dependent on the total component of the magnetic field in the xy-plane, $B_\parallel$. Plots of the concurrence as a function of T for different values of $B_\parallel$, where $B_\perp = 0$, are presented in figure 3(a). For $T = 0$, there is a jump in the concurrence at $B_\parallel = \sqrt{2}J$ when $\gamma, K = 0$. In this case, one obtains

$$C(T=0) = \begin{cases} 1 & \text{for } B_\parallel < \sqrt{2}J, \\ 2/3 & \text{for } B_\parallel = \sqrt{2}J, \\ \dfrac{1-D^2}{1+D^2} & \text{for } B_\parallel > \sqrt{2}J, \end{cases} \tag{10}$$

with $D = (\sqrt{1+(2B_\parallel/J)^2} - 1)/(2B_\parallel/J)$.

For $B = B_\perp\hat{z}$ and $\gamma, K = 0$, it has already been shown in the literature [10] that, when $B_\perp > J$, although there is no entanglement ($C = 0$) at $T = 0$ (also in agreement with equation (9)), the concurrence presents a maximum at finite T. In fact, at $T = 0$, the ground state is the disentangled state $|\downarrow\downarrow\rangle$, that mixes with the maximally entangled ones as T increases. A similar effect is observed for $B = B_\parallel$ and $\gamma, K = 0$ (see figure 3(a)), although in this case $C(T = 0) \neq 0$ consistently with equation (10). For $B_\parallel < \sqrt{2}J$ the concurrence C decreases monotonically with T, whereas for $B_\parallel > \sqrt{2}J$ it presents a local maximum at finite T.

A plot of C versus $|B|$ is shown in figure 3(b) for different orientations, at $T \simeq 0$ and $\gamma, K = 0$. For all orientations there is a jump at $T = 0$. It occurs when

$$2B_\perp^2 + B_\parallel^2 = 2J^2. \tag{11}$$

Differently from the case $B = B_\perp\hat{z}$, if $B_\parallel \neq 0$, then the concurrence falls to a non-null value above the singularity. Thereafter, C tends smoothly to zero with increasing intensity of the magnetic field.

In figure 4 we exhibit T^* as a function of the intensity $|B|$ for different orientations of the field when $\gamma = 0$ and $K = 0$. Notice that if the magnetic field is oriented in the z-direction, then the threshold temperature does not depend on the intensity of the field. However, when the field has a non-null component

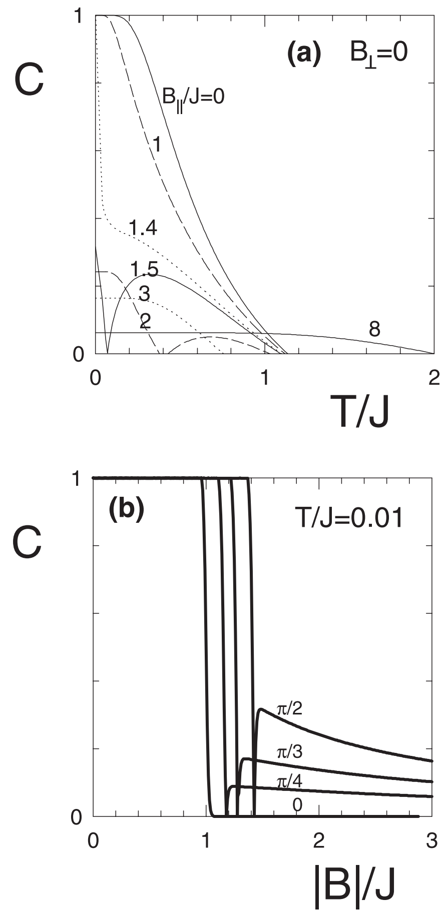

Figure 3. Concurrence C as a function of T (a) for different values of $B_\parallel$ and $B_\perp = 0$ and C as a function of $|B|$ at $T \simeq 0$ and (b) for different orientations of the magnetic field. $B_\perp = |B|\cos\theta$ and $B_\parallel = |B|\sin\theta$, where the values of angle θ are indicated in the figure. Here $\gamma = 0$ and $K = 0$.

in the xy-plane ($B_\parallel \neq 0$), T^* increases for sufficiently large $B_\parallel$. By comparing figures 2 and 4, notice that $B_\parallel \neq 0$ yields an effect qualitatively similar to that due to $B_\perp$ when the system has cross-terms ($K \neq 0$). A similar effect is also observed with $B_\perp$ when the system is anisotropic ($\gamma \neq 0$) [11]. But, even in the absence of anisotropy or cross-terms, the external magnetic field can be used to control the entangled domain by switching on the $\parallel$-component.

4. Conclusions

To summarize, we have investigated the thermal entanglement of two-qubit XY systems with additional exchange cross-terms ($K \neq 0$) as well as with anisotropic couplings ($\gamma \neq 0$). Arbitrary orientations of an external magnetic field B were considered. We have found that, *even in the absence of anisotropy*, cross-terms enable control of the threshold temperature T^*, which defines the domain of entangled states, by varying the intensity of the magnetic field in the z-direction. Moreover, even in the absence of anisotropy or cross-couplings, the degree of entanglement can be manipulated by suitably changing the relative orientation of the sample with respect to a constant magnetic field. As a

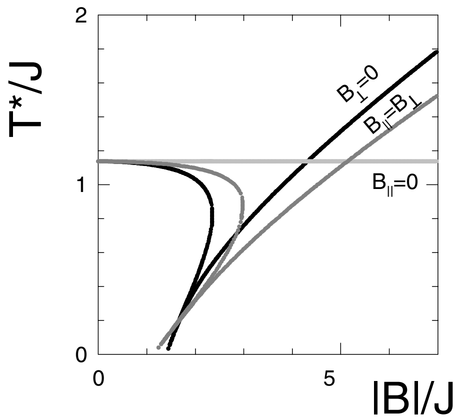

Figure 4. Temperature T^* versus $|B|$ for the orientations of the field indicated in the figure. The entangled domain is the low-temperature one. Here $\gamma = 0$ and $K = 0$.

perspective, it would be interesting to investigate the effect of increasing the number of spins.

Since thermal entanglement is a natural type of entanglement for a system embedded in a thermal environment, we expect that the present results could be useful for solid state applications. In particular we expect that the understanding of thermal entanglement could be exploited within the context of quantum communication and information processing, e.g., along the lines of quantum heat machines [16].

Acknowledgments

AMCS is grateful to C Tsallis for his kind hospitality at CBPF where this work was done. We acknowledge Brazilian Agencies FAPERJ and CNPq for financial support.

References

[1] Platzman P M and Dykman M I 1999 *Science* **284** 1967
[2] Vala J and Whaley B 2002 *Phys. Rev.* A **66** 022304
[3] Loss D and DiVincenzo D P 1998 *Phys. Rev.* A **57** 120
Hu X and DasSarma S 2000 *Phys. Rev.* A **61** 062301
[4] Kane B E 1998 *Nature* **393** 133
[5] Imamoglu A *et al* 1999 *Phys. Rev. Lett.* **83** 4204
[6] Quiroga L and Johnson N F 1999 *Phys. Rev. Lett.* **83** 2270
[7] Zheng S-B and Guo G-C 2000 *Phys. Rev. Lett.* **85** 2392
[8] Mozyrsky D, Privman V and Glasser M L 2001 *Phys. Rev. Lett.* **86** 5112
[9] Raussendorf R and Briegel H J 2001 *Phys. Rev. Lett.* **86** 5188
[10] Wang X 2001 *Phys. Rev.* A **64** 012313
[11] Kamta G L and Starace A F 2002 *Phys. Rev. Lett.* **88** 107901
[12] Dzyaloshinski I 1958 *J. Phys. Chem. Solids* **4** 241
Moriya T 1960 *Phys. Rev.* **120** 91
Burkard G and Loss D 2002 *Phys. Rev. Lett.* **88** 047903
[13] Terhal B M and DiVincenzo D P 2002 *Phys. Rev.* A **65** 032325
[14] Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78** 5022
Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
[15] Arnesen M C, Bose S and Vedral V 2001 *Phys. Rev. Lett.* **87** 017901
[16] Scully M O 2001 *Phys. Rev. Lett.* **87** 220601
Scarani V, Ziman M, Stelmachovic P, Gisin N and Buzek V 2002 *Phys. Rev. Lett.* **88** 097905