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Risk aversion in economic transactions

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Abstract. – Most people are risk-averse (risk-seeking) when they expect to gain (lose). Based on a generalization of "expected utility theory" which takes this into account, we introduce an automaton mimicking the dynamics of economic operations. Each operator is characterized by a parameter q which gauges people's attitude under risky choices; this index q is in fact the entropic one which plays a central role in nonextensive statistical mechanics. Different long-term patterns of average asset redistribution are observed according to the distribution of parameter q (chosen once forever for each operator) and the rules (*e.g.*, the probabilities involved in the gamble and the indebtedness restrictions) governing the values that are exchanged in the transactions. Analytical and numerical results are discussed in terms of how the sensitivity to risk affects the dynamics of economic transactions.

People are sensitive to risk, a characteristic also observed in animals such as rats, birds and honeybees [1]. The usual preference for a sure choice over an alternative of equally or even more favorable expected value is called *risk aversion*. Actually, most people feel aversion to risk when they expect, with moderate or high probability, to gain and attraction to risk when they expect to lose. These tendencies are inverted for very low probabilities [2]. Certainly, this pattern of attitudes affects most human decisions since chance factors are always present, *e.g.*, in medical strategies, in gambling or in economic transactions. In particular, in the context of finances, the attitude of economic operators under risky choices clearly is one of the main ingredients to be kept in mind for realistically modeling market dynamics.

In economics, traditionally, the analysis of decision making under risk was treated through the "expected utility theory" (EUT) [3], on the assumption that individuals make rational choices. More precisely, the *expected value* E, corresponding to the *prospect* $\mathcal{P} \equiv (x_1, p_1; \ldots; x_n, p_n)$ such that the outcome x_i (gain if positive; loss if negative) occurs with probability p_i,

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is given by $E(\mathcal{P}) = \sum_{i=1}^{n} \chi(x_i)p_i$, where the weighting function $\chi(x_i)$ monotonically increases with x_i. (Clearly, a statistically fair game corresponds to $\chi(x_i) = x_i$.) There are, however, aspects of risk sensitivity that are not adequately contemplated within EUT. Such features were exhibited, through experiments with hypothetical choice problems, by Kahneman and Tversky [2]. They then proposed a generalization to EUT equation within "prospect theory" (PT) [2]: $E(\mathcal{P}) = \sum_{i=1}^{n} \chi(x_i)\Pi(p_i)$, where the weighting function $\Pi(p_i)$ monotonically increases with p_i. Its typical shape (corresponding to the most frequent human attitude) is presented in fig. 1, as sketched by Tversky and collaborators [2,4] on the ground of experiments and observations. The weight $\Pi(p)$ is, basically, concave for low and convex for high probabilities, with $\Pi(0) = 0$ and $\Pi(1) = 1$. Moreover, most individuals satisfy i) $\Pi(p)+\Pi(1-p) \leq 1$ (the equality holds for $p = 0, 1$) (*subcertainty*), ii) $\Pi(p\,s)/\Pi(p) \leq \Pi(p\,r\,s)/\Pi(p\,r)$ for $0 < p, r, s \leq 1$ (*subproportionality*), iii) $\Pi(p) < (>) p$ for high (low) probabilities (*under(over)weighting*), and, iv) for very low probabilities, $\Pi(p\,r) > r\Pi(p)$ for $0 < r < 1$ (*subadditivity*). The following functional forms have been proposed [5] in the context of nonextensive statistical mechanics [6]: $\Pi(p) = p^q$ ($q \in \Re$) and $\Pi(p) = p^q/(p^q+(1-p)^q)$, usually referred to as *escort probability*. Other functional forms are also available in the literature [7], such as $\Pi(p) = p^q/[p^q+(1-p)^q]^{1/q}$ and $\Pi(p) = p^q/[p^q + A(1-p)^q]$, where $A > 0$. Clearly, $A = 1$ recovers the escort probability and for appropriate choices of (q, A), the latter expression can satisfy all the properties detailed above. In all these cases, each individual is characterized by a set of parameters which yields a particular $\Pi(p)$ representing the subjective processing that the individual makes of known probabilities p in a chance game.

More recently, PT was generalized [8] using a rank-dependent or cumulative representation where the "decision weight" multiplying the value of each outcome is distinguished from the probability weight. This interesting generalization is, however, irrelevant for the present study. Indeed, we will deal here with simple prospects with a single positive outcome, hence both versions coincide.

In the present work we investigate the consequences of risk-averse attitudes in the dynamics of economic operations. In order to do so, we apply methods of statistical physics. This strategy has proved to be very useful in several previous works [9] (see also [10] for general discussions on the application of statistical physics methods in economy). Here we introduce an automaton simulating monetary transactions among operators with different attitudes under risky choices. Elementary operations are of the standard type used in hypothetical choice problems that exhibit risk aversion [2]. By following the time evolution of the asset position of the operators, it will be possible to conclude on the consequences of each particular attitude.

We will restrict our study to the regime of moderate and high probabilities where most people are risk-averse for gains. In this regime, human behavior can be satisfactorily described by the weighting function $\Pi(p) = p^q$. This expression, which has a simpler form than other weights describing the full domain, is the one adopted in the present work. Furthermore, since we will focus on probabilistic factors, we assume that all individuals have the simplest utility function, namely $\chi(x) = x$. Adopting these choices and, additionally, assuming that the probability weight $\Pi(p)$ is the same for gains and losses, a unique parameter (q) characterizes each individual according to the attitude under risky choices. An ideally rational individual has $q = 1$ while most individuals "feel" probabilities with $q > 1$.

The present automaton simulates monetary transactions among N operators. We assume for simplicity that in each elementary transaction two agents participate. One of them proposes to the other a choice between two alternative ways of either paying money or of receiving money. As an illustration of the former case, the proponent typically asks: "What do you prefer: to receive a *certain* quantity X or to play a game where you receive Y with probability P, such that $PY = X$, and nothing with probability $1 - P$?". More precisely, the

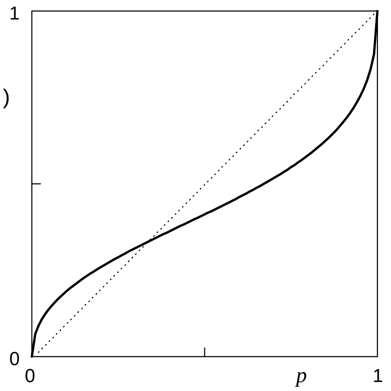

Fig. 1

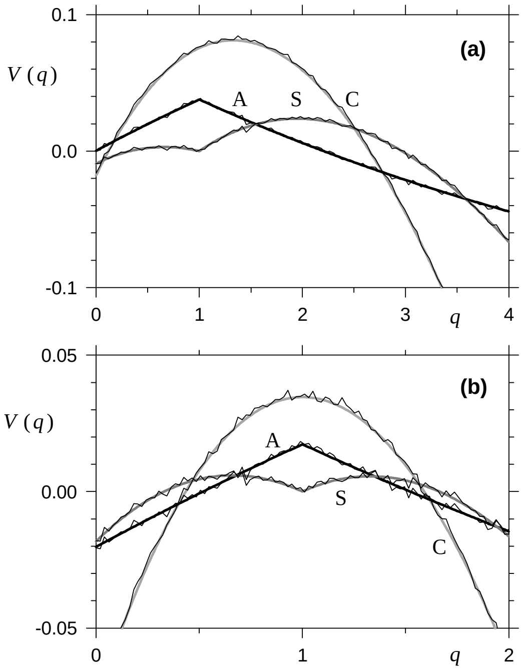

Fig. 2

Fig. 1 – Typical shape of the function $\pi(p)$. The dotted line corresponds to the usual expectation value (EUT).

Fig. 2 – $V(q)$ *vs.* q, obtained from eq. (3), for cases A, S and C and two different initial distributions of parameter q: uniform in $[0,4]$ (a), and uniform in $[0,2]$ (b). Without debt restraints V gives the average time evolution of assets: $\overline{M}(q,t)-\overline{M}(q,0)=StV(q)/N$. Thin lines correspond to the average over 10^4 numerical simulations with $M(q,0)=M_0=1000$, $N=100$, $S=10$, $P=0.85$ and $t=10^4$.

alternative choices are: 1) a *certain* prospect $\mathcal{P}=(\pm\alpha S,1)$, with $\alpha>0$, and 2) a *risky* one $\mathcal{P}=(\pm S,P;0,1-P)$, where S is a positive quota and the $+/-$ apply to the cases when the proponent receives/pays. The value of α depends on the psychological factors q of both operators in a way that will be defined below for each one of the models conceived. The also possible case where the choice is between a *fixed* certain prospect $\mathcal{P}=(\pm S,1)$ and a *variable* risky one $\mathcal{P}=(\pm\alpha S,P;0,1-P)$ will not be analyzed here since it yields quite analogous results. Along the dynamics, the probability P and the quota S, the two parameters characterizing the elementary operations, are kept fixed. Clearly, the present games are not the kind of operations that actually occur in a financial market. However, in the sense of the theory of financial decisions, they illustrate the risk aversion phenomenon.

Let us consider N agents with values of q uniformly distributed in the interval $[Q_1,Q_2]$. For simplicity we take $q_k=Q_1+(Q_2-Q_1)(k-1)/(N-1)$ with $k=1,\ldots,N$. Each operator k has an initial amount of money $M_k(0)$. In each time step, a transaction between the agents of a randomly chosen ordered pair (i,j) occurs. The exchanged quantity is taken positive if the proponent (i) is the one who receives the money and negative otherwise. Whether the exchanged quantity is positive (i receives the money) or negative (j receives the money), it is randomly drawn at each step of the dynamics.

Restrictions on the level of indebtedness of the operators can be imposed. We consider three cases (NR, OR and PR): In case NR, there are *no restraints* and operators can be indebted without limit; in OR, there are *opportunistic* restraints and agents can also operate indefinitely except that they do not pay when they would have to do so if at a given step of the dynamics their assets become less than a minimal quantity M^* (*i.e.*, operators can become swindlers occasionally); finally, in PR, agents become *permanently* forbidden to trade from the instant when their assets fall below the threshold M^*. This threshold introduces nonlinearity into the problem.

For each one of these three cases, other variations (A, S and C) were considered following different definitions of α: in A (alter-referential), operator i, the proponent, somehow knows the psychology of j (q_j); in S (self-referential), operator i ignores q_j and hence attributes to j her/his own value of q; finally, in C (consensual), operators i and j act by consensus.

In variations A and S, the proponent, i, will present, according to the hypothesis made on the q of j (q_j^{hyp} taken to be q_j for model A and q_i for model S), a certain alternative which i considers the worst between the one which has the same standard expected value as the game ($\pm PS$) and the one that i believes to have for j the same expectation as the game ($\pm P^{q_j^{\mathrm{hyp}}} S$). This opportunistic behavior can be expressed through the factor α, as presented below:

– Case A: $\alpha = \min(\max)\{P, P^{q_j}\}$ if i pays(takes).

– Case S: $\alpha = \min(\max)\{P, P^{q_i}\}$ if i pays(takes).

Note that, in a statistically fair game, it should be $\alpha = P$ so that both prospects (the certain and the risky ones) would have the same standard expected value. On the other hand, agent j will choose the risky prospect either 1) if $P^{q_j} > \alpha$ when i pays or 2) if $P^{q_j} < \alpha$ when i takes or 3) in 50% of the cases if $P^{q_j} = \alpha$.

– Case C: In this instance, there is not a proponent and both players have a symmetric role arriving at a consensus. Hence, let us consider, without loss of generality, the case when i pays and j takes. If $q_i < q_j$, there is agreement for the risky alternative, otherwise, the agreement is for the certain choice with α an intermediate value between P^{q_i} and P^{q_j}, for instance, as we will adopt, $\alpha = (P^{q_i} + P^{q_j})/2$.

Combining all these possibilities, we have a total of nine models: (NR, A), (NR, S), ..., (PR, C).

Let us discuss first the models of the type (NR, *), for which there is no indebtedness restriction. The amount of money of operator i at time t, $M_i(t)$, *on average* over a large number of realizations (histories), is given by

$$\overline{M_i}(t) = \overline{M_i}(0) + \frac{t}{N}\, S\, V_i\,, \quad \forall i, t, \tag{1}$$

where $V_i = \frac{1}{N}\sum_j (G_{ij} - G_{ji})$, such that the G matrix has nonnegative elements. For instance, for case (NR, C), explicitly one has

$$G_{ij}^{C} = \begin{cases} \dfrac{1}{2}(P^{q_i} + P^{q_j}), & j < i, \\ 0, & j = i, \\ P, & j > i. \end{cases} \tag{2}$$

The continuum approximation of V_i for all models (NR, *), considering a uniform distribution of the parameter q in the interval $[Q_1, Q_2]$, results to be

$$V(q) \simeq \frac{1}{Q_2 - Q_1} \times$$
$$\times \begin{cases} \frac{1}{4}\Big((Q_2+Q_1-2)P + \frac{2P - P^{Q_1} - P^{Q_2}}{\ln P} + (Q_2-Q_1)(P-P^q)\,\mathrm{sign}(1-q)\Big) & \text{(A)}, \\ \frac{1}{2}(1-q)P + \frac{1}{2}\frac{P^q - P}{\ln P} - \frac{1}{2}(P-P^q) \times \begin{cases} (q-Q_1)\,, & \text{if } q<1 \\ (q-Q_2)\,, & \text{otherwise} \end{cases} & \text{(S)}, \\ (Q_2+Q_1-2q)\Big(P - \frac{1}{2}P^q\Big) + \frac{2P^q - P^{Q_1} - P^{Q_2}}{2\ln P} & \text{(C)}. \end{cases} \quad (3)$$

$V(q)$ *vs.* q for models (NR,*) is illustrated in fig. 2. $V(q)$ rules the average evolution of assets, being

$$\overline{M}(q,t) = \overline{M}(q,0) + \frac{t}{N} SV(q)\,, \quad \forall q, t. \quad (4)$$

From fig. 2a, corresponding to a more realistic distribution of q than fig. 2b (since about 75% of the people are risk-averse when high probabilities are involved), note that for (NR, A) (where the proponent knows the psychology of the other) the maximum emerges at $q = 1$ (the rational player), while for (NR, S) (where the proponent acts following the own psychology) we observe the emergence of a minimum at $q = 1$ and of two maxima on both sides of $q = 1$, the absolute maximum being the one for $q > 1$ (agents who are conservative for gains). Case (NR, C) (consensus) corresponds to an intermediate situation where the maximum asset increase occurs for q above 1.

For models (OR, *) and (PR, *) the average time evolution of M is not linear with t. In these cases, the evolution follows eq. (1) (or its continuous version eq. (4)) up to time τ when some agent's asset falls below the threshold M^*. In models (OR, *), from that instant on, the discrete Markovian automaton can be described, on average, by

$$\overline{M_i}(t+1) = \overline{M_i}(t) + \frac{S}{N^2}\sum_j (\overline{H_j} G_{ij} - \overline{H_i} G_{ji})\,, \quad \forall i,\ t > \tau, \quad (5)$$

where $H_k \equiv H(M_k(t) - M^*)$ is a Heaviside function. $\overline{M_i}(t)$ evolves to a nontrivial steady state whose extrema coincid with those of V_i. This steady state M_k^{SS} is approximately of the form $M_k^{\mathrm{SS}} = a/(b + V^{\max} - V_k)$, $\forall k$, where $a, b > 0$. Its continuous version is the long-time solution of the nonlinear diffusion equation $\partial_t M(q,t) = \partial_q(M(q,t)\partial_q V(q)) + a\partial_{qq} \ln M(q,t)$. In fact, $(-V(q))$ acts as the potential of an effective drift which rules the dynamics. In fig. 3, $\underline{M(q,t)}$, averaged over several realizations, *vs.* q, is illustrated for models (OR, *). The maxima of $\overline{M}$, $\overline{M}_{\max}$, and its corresponding q, $q_{\max}$, are plotted as a function of time t in figs. 3b and c to control the stability of the resulting state. The plots are invariant when all monetary amounts (*i.e.*, M_0, M^* and S) are multiplied by a common factor. As either M^*/M_0 or S/M_0 increase, the steady state broadens. Increasing S shortens the time scale and increases the amplitude of fluctuations. Although in the example $M(q,0) = M_0$, the steady state does not depend on the initial distribution of money.

For models (PR, *), the dynamics follows a different evolution. $\overline{M}(q,t)$ evolves to a Dirac δ-function centered at $q = 1$ or at the boundary closer to $q = 1$ (see fig. 4). As for the (OR, *) models, the stability is controlled by watching $\overline{M}_{\max}$ and $q_{\max}$ *vs.* t. Note that in all cases $q_{\max} \to 1$, although for case (PR, S) the convergence is slower. Therefore, with this kind of

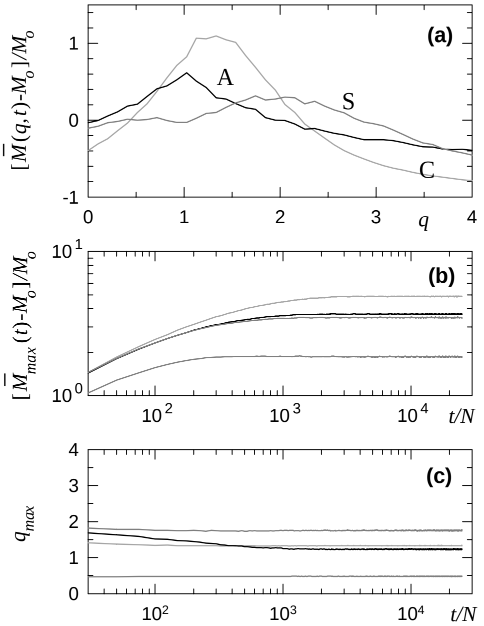

Fig. 3

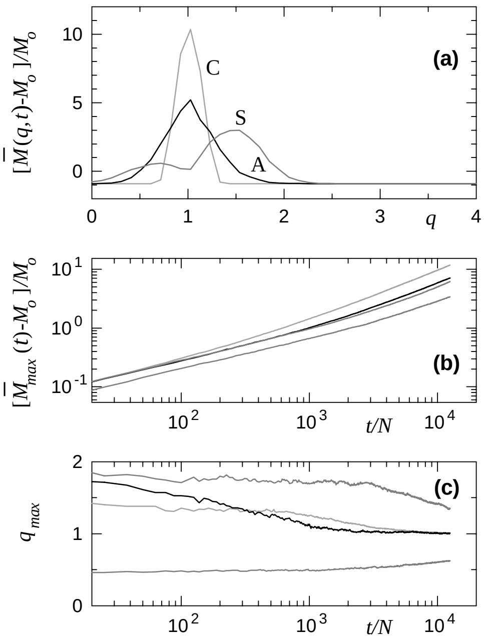

Fig. 4

Fig. 3 – Time evolution of assets with indebtedness restraints of the kind (OR, *) (without exclusion) with threshold $M^* = 100$. (a) $(\overline{M}(q,t) - M_0)/M_0$ *vs.* q at term $t/N = 25000$ when the steady state is already attained, for cases A (black), S (dark gray) and C (light gray). Lines correspond to simulations averaged on 2×10^3 experiments with $M(q,0) = M_0 = 1000$, $N = 40$, $S = 100$ and $P = 0.85$. The plots do not depend on N and depend on (M^*, M_0) only through their ratio. Increasing S shortens the time scale and increases the fluctuation amplitude. (b) $\overline{M}(q_{\max}, t)$ *vs.* t and (c) $q_{\max}$ *vs.* t, where $q_{\max}$ maximizes $\overline{M}(q,t)$. For case S, both maxima are plotted.

Fig. 4 – Time evolution of assets with indebtedness restriction of kind (PR, *) (with exclusion) with threshold $M^* = 100$. (a) $(\overline{M}(q,t) - M_0)/M_0$ *vs.* q at term $t/N = 12500$, for cases A (black), S (dark gray) and C (light gray). Lines correspond to simulations averaged on 10^3 experiments with $M(q,0) = M_0 = 1000$, $N = 40$, $S = 10$ and $P = 0.85$. (b) $\overline{M}(q_{\max}, t)$ *vs.* t and (c) $q_{\max}$ *vs.* t, where $q_{\max}$ maximizes $\overline{M}(q,t)$. For case S, both maxima are plotted.

indebtedness restraint, the more rational operator wins at the long term. Here the final state does not depend on the distribution of q, which will just generate a different $V(q)$, as soon as it contains $q = 1$ (if not, the boundary closer to $q = 1$ wins). On the other hand, the final state depends on the initial distribution of assets once some individuals may have assets below the threshold and are forbidden to play from the beginning.

On average, the rational player (with $q = 1$) wins from every other ($G_{uj} - G_{ju} \geq 0$, $\forall j$ and $q_u = 1$). That is why with restraints of type PR such player is the long-term winner. Operators with extreme values of q are long-term losers. With indebtedness restrictions of the kind PR, they have to abandon the game. With restrictions of the kind OR (cheating is not forbidden), they are allowed to remain in the game, but their assets keep fluctuating close to the threshold M^*. A nontrivial steady state appears in this case. Operators with q_k a bit above one (a bit risk averse for gains) lose from some neighbors but win more in the total sum, *i.e.*, $\sum_j (G_{kj} - G_{jk}) \geq 0$), as soon as those with extreme values of q remain in the game. Consequently, the maximum of the steady state is located at q above one (primacy of the conservatives). In case (OR, *) the initial distribution of parameter q will affect the form of the function V and therefore the shape of the steady state governed by the effective potential $(-V)$. On the other hand, the steady state does not depend on the initial distribution of money.

In conclusion, the type of conditions limiting indebtedness is critical for defining the nature of the long-term evolution, *i.e.*, existence or not of a nontrivial steady state. The details of this steady state depend, among other factors, on the distribution of the parameter q of the operators. One also observes that the final state is invariant under initial redistribution of money. Paradoxically enough, some level of cheating avoids extreme wealth inequality to become the stationary state. However, one must keep in mind that the distribution of q is kept fixed along the dynamics and, therefore, the psychological effect of asset position is not taken into account in the present model. Such dynamics would provide an improved, more realistic model. In fact, a model which, in addition to this learning-from-experience dynamics, would use a weight such as that of fig. 1 simultaneously with an appropriate nonlinear utility function, might very well constitute a quite realistic frame for taking into account the well-known human risk aversion in the context of collective trading.

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