Breakdown of Exponential Sensitivity to Initial Conditions: Role of the Range of Interactions

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(Received 22 January 1998)

Within a microcanonical scenario we numerically study an $N$-sized linear chain classical inertial XY model including ferromagnetic couplings which decrease with distance as $r^{-\alpha}$ ($\alpha \geq 0$). We show that for $N \to \infty$ (thermodynamic limit): (i) The energy per particle $E_N/N$ scales like $N^\alpha = (N^{1-\alpha} - 1)/(1 - \alpha)$; (ii) The properly scaled maximum Lyapunov exponent $\lambda_{\text{max}}^N$ tends, for $E_N/(NN^*)$ above a threshold, to zero for and only for $\alpha \leq 1$. These results are analogous to those observed in low-dimensional and in self-organized critical dissipative systems. This entire picture suggests a connection with the nonextensive thermostatistics recently introduced by one of us.

PACS numbers: 05.45.+b, 05.20.–y, 05.70.Ce, 64.60.Lx

\begin{equation}
\mathcal{H} = \frac{1}{2} \sum_{i=1}^{N} L_i^2 + \frac{1}{2} \sum_{i \neq j}^{N} \left(1 - \cos(\theta_i - \theta_j)\right) r_{ij}^\alpha
\end{equation}

\begin{equation}
= E_k + E_P \quad (\alpha \geq 0; r_{ij} = 1, 2, 3, \ldots),
\end{equation}

where, without loss of generality, we have considered unit momenta of inertia and unit first-neighbor coupling constants, and where $\{\theta_i, L_i\}$ are conjugate canonical pairs. Since the model is essentially defined on an $N$-sized ring, every pair of (classical) spins determines, along the ring, two “distances” and not one: the $r_{ij}$ which we consider in the Hamiltonian is the minimal one. The model basically is a classical inertial XY ferromagnet (coupled rotators), and the limiting cases $\alpha \to \infty$ and $\alpha = 0$ correspond to the first-neighbor [1,16] and mean-field-like models, respectively. The kinetic contribution $E_k$ contains $N$ different terms; the potential one $E_P$ contains $N(N - 1)/2$ different terms and $E_P/N^*$ is bounded from above by a value which, for the more general, $d$-dimensional, case is asymptotically proportional to $N^* \equiv \int_1^{N^{d/4}} \, dr \, r^{d-1} \, r^{-\alpha} = \frac{N^{1-\alpha/2} - 1}{1 - \frac{\alpha}{2}}$. In the $N \to \infty$ limit, $N^*$ behaves, respectively, like $\frac{N^{1-\alpha/2}}{1 - \frac{\alpha}{2}}$ (hence $N^*$ for $\alpha = 0$, $\ln N$, and $\frac{1}{\beta}$ for $0 \leq \alpha < d$, $\alpha = d$, and $\alpha > d$.

It is clear that the model is thermodynamically extensive for $\alpha > d$ and nonextensive otherwise. However, this model can be written in an artificially pseudoextensive manner as follows:

\begin{equation}
\mathcal{H}' = \frac{1}{2} \sum_{i=1}^{N} L_i^2 + \frac{1}{2N^*} \sum_{i \neq j}^{N} \left(1 - \cos(\theta_i - \theta_j)\right) r_{ij}^\alpha
\end{equation}

\begin{equation}
(\alpha \geq 0; r_{ij} = 1, 2, 3, \ldots).
\end{equation}

This presentation should be considered as very artificial indeed, since it turns the microscopic coupling constants $N$ dependent, i.e., modified through macroscopic information. At this (conceptually rather high) price, we obtain a thermodynamically (pseudo) extensive quantity for
all values of $\alpha$ [17]. In particular, for $\alpha = 0$, we obtain the usual mean-field-like form, within which the coupling constant is renormalized by $N$. This special case has been focused by several authors [18,19]. Recently, Latora, Rapisarda, and Ruffo [19] obtained quite interesting results which we shall connect to ours later on.

Before going on let us exhibit a convenient connection between $H$ and $H'$. If we take into account that the variables $h_{Li}$ involve a first derivative with respect to time $t$, we immediately verify that

$$H = N H',$$

where the time scales $t$ and $t'$ respectively, associated with $H$ and $H'$ satisfy $t' = \sqrt{N} t$. Consequently, if we recall that by definition the Lyapunov exponents appear multiplied by $t$ in all expressions concerning the sensitivity to the initial conditions, and note that $\lambda^\text{max}_N$ and $\lambda^\text{max}_{N'}$ are the largest Lyapunov exponents, respectively, associated with Hamiltonians $H$ and $H'$, we have that

$$\lambda^\text{max}_N = \lambda^\text{max}_{N'} t.$$

This relationship will prove generically helpful later on (and very particularly in order to compare our $\alpha = 0$ results with those presented in [19]). At the present stage let us make clearer what we refer to. If we call $x$ the direction in the $2N$-dimensional phase space corresponding to the largest Lyapunov exponent and define $\xi = \lim_{\Delta \to 0} \frac{\Delta x(t)}{\Delta t}$, then we have $\xi = \exp(\lambda^\text{max}_N t)$ for the Hamiltonian $H$. It is, however, known that Lyapunov exponents, due to the fact that they carry a physical dimension (inverse time), are not mathematically fully defined in the sense that they are not dimensionless quantities. Consequently, either we have to restrict our considerations to ratios of Lyapunov exponents, or we must refer them all to a unique conventional time unit. We shall adopt the latter. More precisely, we can rewrite $\xi$ as follows: $\xi = \exp(\lambda^\text{max}_N t) = \exp([\lambda^\text{max}_N / \sqrt{N}] t) = \exp(\lambda^\text{max}_{N'} t)$. The practical use of this transformation will become transparent on Fig. 1.

The numerical simulation we have implemented in the present work is a standard molecular dynamics one within a microcanonical scenario (applied to Hamiltonian $H$ associated with $N$ spins), this is to say at fixed total energy $E_N$. As initial conditions ($t = 0$ values) we have used random values of $\theta_i$ compatible with the expected equilibrium distribution (which, for the relatively high energy region mainly focused in the present Letter, corresponds to a uniform distribution in the interval $[0, 2\pi]$), and a “water-bag” distribution (i.e., a symmetric uniform distribution).

![FIG. 1](image-url)

FIG. 1. Evolution of $E_k / E_N$ as a function of $t$, for a $N = 30$ single realization for typical values of ($\alpha, N$) (the insets contain the same examples as functions of $t$). We recall that, for $\alpha > 1$, $N'$ is asymptotically independent from $N$, whereas, for $\alpha \leq 1$, $N'$ strongly depends on $N$. It is remarkable that the dominating frequency, in the $t$ variable, of the fluctuations is $N$ independent $\forall \alpha$, whereas, in the $t$ variable, this occurs only for $\alpha > 1$. 5314
distribution on a compact support, in such a way that the total angular momentum equals zero) for the $\{L_i\}$. The time evolution has then followed Newton’s law (using a fourth-order symplectic algorithm [20] with a relative error in the total energy conservation less than $10^{-4}$). A typical evolution of $E_k$ is presented in Fig. 1. For each choice of ($\alpha, E_{Nt}, N$) we have typically run 100 realizations for small systems ($N = 5, 10$) down to a few for large systems ($N = 1000$), and then averaged, over all the realizations, the maximal Lyapunov exponents (calculated by the method of Benettin et al. [21]). Typical results are presented in Figs. 2 and 3. The fact that, for fixed ($\alpha, N$), the thermodynamic variable which generically emerges [as follows from our earlier considerations concerning Eq. (1)] is $\frac{E_N}{NN^*}$ and not the usual $\frac{E_N}{N}$, one clearly reflects the possible nonextensivity of the system: only for $\alpha > 1$ (short-range interactions) is $\frac{E_N}{N}$ the convenient variable. Also, since both $\frac{E_N}{NN^*}$ and $\frac{E_{Nt}}{N}$ remain finite in the $N \rightarrow \infty$ limit, so does $\frac{E_{Nt}}{NN^*}$ which is $\sim \frac{\lambda_{\alpha}}{N^2}$; hence, for $\alpha \leq 1$, the correct variable to describe, say, an equation of states is not the usual intensive variable $T$ but the present renormalized one [22], i.e., $T^* \equiv \frac{T}{N^2}$ (through these transformations for $\alpha = 0$, we precisely recover Fig. 1 of [19], for instance). These facts are absolutely consistent with recent results presented in the literature for a variety of long-range interaction physical situations [22] (Lennard-Jones-like fluids, magnets).

Let us summarize our main results. We have shown, for the particular classical magnetic model herein studied, that the thermodynamics is extensive if and only if $\alpha > 1$ (short-range interactions); the nonextensivity which appears for $0 \leq \alpha \leq 1$ (long-range interactions) is illustrated, among others, by the fact that the macroscopic energy quantity which remains well defined at the thermodynamic limit ($N \rightarrow \infty$) is $\frac{E_N}{NN^*}$ and not the usual one $\frac{E_{Nt}}{N}$. The microscopic dynamical counterpart of this extensive-nonextensive critical point is very enlightening, namely, the fact that, above a ($\alpha$-dependent) threshold of $\frac{E_{Nt}}{NN^*}$ (approximately corresponding to the maxima of $\lambda_{\alpha}^{\text{max}}$ in the plots of Fig. 2), the largest properly scaled Lyapunov exponent ($\lambda_{\alpha}^{\text{max}}$) remains, at the thermodynamic limit, positive and finite for short-range interactions but collapses to zero (and with it the entire Lyapunov spectrum) for long-range interactions (see Fig. 3). In other words, the range of the interactions controls the type of sensitivity to the initial conditions that a large system will exhibit: strong chaos (exponential law) clearly is the case for short-range interactions, and weak chaos (presumably power law) for long-range interactions. The former will exhibit standard ergodicity, whereas important anomalies are to be expected for the latter. It goes without saying that the present scenario is expected to hold for large classes of Hamiltonian systems and not only for the system herein studied.

**FIG. 2.** The $\frac{E_N}{NN^*}$ dependence of the properly scaled Lyapunov exponent $\lambda_{\alpha}^{\text{max}}$ for $\alpha = 1.5$ (a) and $\alpha = 0.2$ (b) and typical values of $N$. As it is illustrated in Fig. 3, the $N \rightarrow \infty$ limit yields, for high enough energies (essentially above the paraferro phase transition critical value), a nonvanishing (vanishing) $\lambda_{\alpha}^{\text{max}}$ for $\alpha > 1 (\leq 1)$.

**FIG. 3.** $\lambda_{\alpha}^{\text{max}}$ versus $N$ (log-log plot) for typical values of $\alpha$ and $\frac{E_{Nt}}{NN^*} = 5$. The full lines are the best fittings with the forms $(a - b \ln N)/(N^\gamma)$. Consequently, $\lambda_{\alpha}^{\text{max}} \propto N^{-\kappa(\alpha)}$ where $\kappa(\alpha) = (1 - \alpha)c$, for $0 \leq \alpha < 1$ and $\kappa(\alpha) = 0$ for $\alpha > 1$; for $\alpha = 1$, $\lambda_{\alpha}^{\text{max}}$ is expected to vanish as a power of $1/\ln N$. Inset: $\kappa$ versus $\alpha$ (related random matrices arguments will be detailed elsewhere).
To discuss the present (as well as other) anomalies, a
generalized statistical formalism has been advanced and
developed during the last few years [15]. The general-
ization essentially consists in considering the following
entropic form: \( S_q = \frac{1}{q-1} \sum_{i=1}^{\infty} p_i^q \) (\( \sum_{i} p_i = 1 \); \( q \in \mathbb{R} \) character-
izes the degree of nonextensivity), which recovers the usual BG entropy \( (\sum_i p_i \ln p_i) \) in the limit \( q \to 1 \);
\( S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \), if
\( A \) and \( B \) are independent systems in the sense that the
probabilities of \( A + B \) factorize into those of \( A \) and of
\( B \). A wealth of works has shown that the above nonex-
tensive thermostatistical prescription retains much of the
formal structure of the standard theory such as Legendre
thermodynamic structure, \( H \) theorem, Onsager reciprocity
theorem, Kramers and Wannier relations, Bogolyubov
inequality, and thermodynamic stability, among others
[15,23]. This formalism has proved to correctly describe
a variety of dissipative nonlinear systems (both low [12]
and high [14] dimensional cases, respectively, related
to the edge of chaos and to self-organized criticality).
Details can be seen in [12], but, in particular, it has been
argued that the solution of \( d\xi/dt = \lambda_q \xi^q \) is given by
\( \xi(t) = [1 + (1 - q)\lambda_q t]^{1/(1-q)} \). This solution recovers
the standard, exponential law in the limit \( q \to 1 \) (exten-
sive case), but it implies a power-law sensitivity when
nonextensivity takes place \( (q \neq 1) \). Several examples of
\( q \neq 1 \) situations have been exhibited [12] (logisticlike,
periodic, and circular maps as well as in the Bak-Sneppen
model for biological evolution). The attractors towards
which the systems evolve are complex (multifractal)
one, a fact which implies anomalies in what concerns
the validity of standard ergodicity. The present work
provides evidence that Hamiltonian (i.e., conservative)
long-range interaction systems might be further examples
of nonextensivity along the above lines \( (q \neq 1) \).

We acknowledge fruitful remarks from G. Casati,
E. M. F. Curado, A. R. Plastino, and R. O. Vallejos. This
work was partially supported by FAPERJ, CNPq, and
PRONEX (Brazilian Agencies).

[17] For instance, in neutral plasma (e.g., an electron gas in a
fixed positive background), where positive and negative
charges coexist, the effective range of the interactions is
normally reduced by local screening. Consequently,
the interactions become short ranged and Eq. (2) is indeed
valid. However, when dealing with confined non-neutral
plasma (see [8]), long-range interactions cannot be masked and
Eq. (2) becomes unrealistic.