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TWO-DIMENSIONAL TURBULENCE IN PURE-ELECTRON PLASMA: A NONEXTENSIVE THERMOSTATISTICAL DESCRIPTION

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Abstract

Huang and Driscoll (1994) studied the two-dimensional turbulent metaequilibrium state that appears in an experiment in which pure-electron plasma evolves in the interior of a conducting cylinder (of radius R_w) in the presence of an external axial magnetic field. They measured the electron radial distribution and compared their data with the profiles resulting from four different phenomenological theories developed by themselves. Two among these theories are based on the optimization of the standard (Boltzmann-Gibbs) entropy, the other two being based on the optimization of the enstrophy. Only one of the latter (Restricted Minimum Enstrophy theory, where *restricted* stands for the fact that a cut-off radius $R_c < R_w$, such that the electron density vanishes for radius $R \geq R_c$, was heuristically introduced in the calculation) survived successfully to the comparison. Then, Boghosian (1996) showed that the RME theory is strictly equivalent (with the correct cut-off naturally emerg-

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ing within the formalism) to the $q = 1/2$ case of the non-extensive thermostatistics recently introduced by one of us, where q is an entropic index characterizing the degree of non-extensivity (the particular case $q = 1$ recovers Botzmann-Gibbs thermostatistics). Along Boghosian's lines we discuss herein the non-extensivity theory for $0 \leq q \leq 1$ and compare it with the Huang and Driscoll data. Our analysis excludes $q = 1$ (in agreement with Huang and Driscoll conclusion) and shows that q slightly above $1/2$ compares with the data not worse than $q = 1/2$. Our evidence partially relies on the fact that the electron radial density is shown to be, as $R \to R_c - 0$, proportional to $(R_c - R)^{\frac{q}{1-q}}$, hence its derivative with respect to R is zero, finite or infinite for $q > 1/2$, $q = 1/2$ and $q < 1/2$, respectively.

Key-words: Two-dimensional turbulence; Electron plasma; Enstrophy; Generalized thermostatistics.

1 Introduction

It is long known that Boltzmann-Gibbs (BG) statistics fails in describing nonextensive systems, e.g., those including long-range interactions or long-range microscopic memory or a fractal relevant phase space (see [1-3] and references therein). In order to overcome this limitation and be able to deal with systems where nonextensivity can not be neglected, an extension of BG thermostatistics was recently proposed[4] and shown to display interesting mathematical properties that make it a consistent formalism[5-9]. This new approach has proved to be useful in describing self-gravitation[10-13], cosmological background radiation[14], solar neutrino problem[15], Lévy-like[16-18] and correlated-like[19, 20] anomalous diffusions, peculiar velocity function of galaxy clusters[21]. Furthermore, it is the basis for the improvement of simulated annealing and related optimization procedures[22-27].

One case where the BG treatment appears to be limited is in the description of a persistent metaequilibrium state experimentally observed in pure-electron plasma[28]. The BG formalism offers only a poor description of the phenomenon whereas it comes out naturally[13] within the framework of nonextensive thermostatistics. It is the aim of the present paper to analyze some aspects of this alternative description.

1.1 The experiment

Huang and Driscoll observed that turbulent pure-electron plasma confined in a conductor cylinder (radius $R_w \simeq 3$ cm and voltage $V \simeq -150$ V) and subject to an uniform axial magnetic field ($B_z \simeq 507$ G) reaches, after a transient, an axisymmetric *metaequilibrium* state (MES) (see [28] for experimental details). Besides carefully measuring the electron density n as a function of the radial coordinate, they described the observed profile by means of a variational treatment considering the constraints of constant mass (number of electrons), energy and angular momentum of the plasma.

1.2 Huang and Driscoll variational descriptions of the MES

Let us define the characteristic quantities $n^\star \equiv N_L/R_w^2$ and $\phi^\star \equiv eN_L$, where N_L is the number of electrons per unit length, e is the electron charge and the scaled radial coordinate is $r \equiv R/R_w$, with R in cm. Moreover, considering the symmetries of the problem, it can be treated as two-dimensional in the plane (r, θ). Then, following Huang and Driscoll, the conserved quantities (i.e., $n^\star$, the dimensionless angular momentum L and the dimensionless energy U) are expressed as

$$
\begin{aligned}
n^\star &= \int n \mathrm{d}^2\mathbf{r}, \\
L &= \int r^2 (n/n^\star) \mathrm{d}^2\mathbf{r}, \\
U &= -\tfrac{1}{2} \int (\phi/\phi^\star)(n/n^\star) \mathrm{d}^2\mathbf{r},
\end{aligned}
\tag{1}
$$

where ϕ is the electrostatic potential and $\mathrm{d}^2\mathbf{r} = r\mathrm{d}r\,\mathrm{d}\theta$.

In order to describe the MES profiles, Huang and Driscoll considered four different phenomenological theories, namely, the point vortex maximum entropy (PV), the continuous fluid maximum entropy (CF), the point vortex global minimum enstrophy (GME) and, finally, the point vortex restricted minimum enstrophy (RME) ones, which we briefly describe now. Let us however anticipate that only the RME one will satisfactorily reproduce their own experimental data. All four theories rely on variational principles, namely, either the maximization of the standard entropy, for the PV and the CF models, or the minimization of the *enstrophy*, for the GME and RME ones, in all cases under constraints (1).

The fact that the maximization of Boltzmann entropy

$$S \equiv -\int (n/n^*) \ln(n/n^*) \mathrm{d}^2\mathbf{r}, \tag{2}$$

either considering the point vortex model or the continuous fluid one, leads to profiles substantially different from the experimental one suggests that the turbulence has not mixed ergodically[28].

Alternatively, minimization of the *enstrophy*

$$Z \equiv \tfrac{1}{2}\int (n/n^*)^2 \mathrm{d}^2\mathbf{r}, \tag{3}$$

under the same constraints, leads to profiles much closer to those experimentally observed. However, negative (i.e., physically unacceptable) densities arise in the GME theory. To overcome this difficulty, Huang and Driscoll imposed a cut-off radius $r_c < 1$, leading to what they call Restricted Minimum Enstrophy model. Although this model satisfactorily describes the observed profiles, there is no clear justification why enstrophy is the quantity to be optimized. Moreover, the introduction of the radial cut-off stands like an expeditious recipe.

1.3 Generalized thermostatistics

The nonextensive thermostatistical proposal is based on the definition of a generalized entropic form S_q characterized by a parameter q related to the degree of non-extensivity. S_q is defined[4] as

$$S_q[f] \equiv \frac{1}{q-1} \int f(\mathbf{x},\mathbf{p})\Big(1 - [hf(\mathbf{x},\mathbf{p})]^{q-1}\Big)\,\mathrm{d}\Omega, \tag{4}$$

where h is a positive constant with the appropriate dimensions, and $f(\mathbf{x},\mathbf{p})$ is a probability distribution in phase space, fulfilling the normalization condition

$$\int f(\mathbf{x},\mathbf{p})\,\mathrm{d}\Omega = 1. \tag{5}$$

Since classical N-particle systems are being considered here, $\mathrm{d}\Omega$ is the volume element in the $6N$-dimensional phase space and $\mathbf{x}$ and $\mathbf{p}$ are $3N$-dimensional vectors representing the coordinates and conjugate momenta of the N particles. Consistently, h has the dimensions of $(length \;\times\; momentum)^{3N}$, i.e., dimensions of $(Planck\ constant)^{3N}$.

Note that in the limit $q \to 1$, we recover the standard logarithmic entropy

$$S_1[f] \equiv -\int f(\mathbf{x},\mathbf{p}) \ln[hf(\mathbf{x},\mathbf{p})]\mathrm{d}\Omega. \tag{6}$$

Within the formalism, the q-expectation value of a dynamical quantity $A(\mathbf{x},\mathbf{p})$ is defined as

$$\langle A\rangle_q \equiv \int [f(\mathbf{x},\mathbf{p})]^q h^{q-1} A(\mathbf{x},\mathbf{p})\,\mathrm{d}\Omega, \tag{7}$$

which corresponds to the measured value (notice that $\langle A\rangle_q = \langle (hf)^{q-1} A\rangle_1$). Consistently, for the variational problem, the maximization of entropy S_q is subject to constraints expressed as q-expectation values.

2 Alternative variational treatment of the MES

Recently, Boghosian[13] showed that another way of describing the MES, from variational principles, is in terms of the maximization of the generalized entropy $S_{1/2}$, under the same constraints as before but expressed as generalized expectation values. In fact, since the enstrophy Z is simply related to the generalized entropy S_2 ($S_2 = 1-2Z$), it can be easily shown that to minimize the enstrophy with given constraints (usual mean values) is equivalent to maximize $S_{1/2}$ when the conserved quantities are expressed as q-expectation values with $q = 1/2$. Therefore, the same description as in the RME model is obtained when the problem is treated within the generalized thermostatistics if the system were characterized by $q = 1/2$. However, in this case, the cut-off radius *needs not to be imposed* but it comes out *naturally* within this treatment. Let us mention that the fact that expressing the physical constraints as q-expectation values (instead of the usual 1-expectation values) provokes a $q \to 1/q$ transformation is not peculiar of $q = 1/2$ but holds for arbitrary q ([13] and references therein).

Since, in principle, parameter q does not need to take the peculiar value 1/2, we examine, in the present work, the possibility that another value of q could better describe the experimental profiles.

Strictly, the variational treatment consists in optimizing $S_q[f]$ with appropriate constraints in the $(\mathbf{x}, \mathbf{p})$ space, including the normalization condition (5). The distribution $f(\mathbf{x}, \mathbf{p})$ thus obtained must then be integrated over $\mathbf{p}$ as well as over all positions along the axis, and finally projected on a one-particle position space. We shall follow along the *phenomenological* lines of Huang and Driscoll[28] and Boghosian[13], directly focusing $n(r)$. More precisely, the description is reduced in such a way that only integrations over the bidimensional space (r, θ) are left to be done. By defining $g \equiv n/n^*$, the generalized entropy and constraints read

$$S_q[g] \equiv \frac{1}{q-1}\int (g - g^q)\mathrm{d}^2\mathbf{r}, \tag{8}$$

$$\int g^q \mathrm{d}^2\mathbf{r} \quad = \quad 1 \qquad (\textit{mass conservation}),$$

$$\begin{aligned}\int r^2 g^q \mathrm{d}^2\mathbf{r} &= L_q \equiv L \ \ (\textit{angular momentum conservation}),\\ -\tfrac{1}{2}\int \frac{\phi}{\phi^\star} g^q \mathrm{d}^2\mathbf{r} &= U_q \equiv U \ \ (\textit{electrostatic energy conservation}).\end{aligned} \tag{9}$$

Moreover, the scaled electrostatic potential

$$\frac{\phi(r)}{\phi^\star} \equiv \int g^q(r')G(\mathbf{r},\mathbf{r}')\mathrm{d}^2\mathbf{r}, \tag{10}$$

(G is the Green's function of the Poisson problem $\nabla^2 G(\mathbf{r},\mathbf{r}') = 4\pi\delta(\mathbf{r}-\mathbf{r}')$), satisfies

$$\nabla^2 \frac{\phi}{\phi^\star} = 4\pi g^q, \tag{11}$$

The constrained optimization of $S_q[g]$ straightforwardly yields

$$\frac{1 - q\, g_q^{q-1}}{q-1} - \alpha q\, g_q^{q-1} - \lambda q r^2\, g_q^{q-1} + \beta q \frac{\phi_q}{\phi^\star} g_q^{q-1} = 0 \tag{12}$$

or

$$\frac{g_q^{1-q} - q}{q-1} - \alpha q - \lambda q r^2 + \beta q \frac{\phi_q}{\phi^\star} = 0 \tag{13}$$

or, yet, taking the Laplacian of both sides

$$\frac{\nabla^2 g_q^{1-q}}{q-1} - 4\lambda q + 4\pi\beta q g_q^q = 0 \tag{14}$$

which coincides with Boghosian[13] Eq. (22) and can be rewritten as

$$g_q'' - q\frac{(g_q')^2}{g_q} + \frac{g_q'}{r} = g_q^q(B g_q^q - A) \tag{15}$$

where $A \equiv 4q\lambda$ and $B \equiv 4\pi q\beta$.

Alternatively, identifying $\rho_q \equiv g_q^q$, we have

$$\rho_q'' - \frac{2q-1}{q}\frac{(\rho_q')^2}{\rho_q} + \frac{\rho_q'}{r} = q\rho_q^{\frac{2q-1}{q}}(B\rho_q - A) \tag{16}$$

Moreover, from g_q and the boundary condition $\phi_q(r=1)=0$, the scaled electrostatic potential is

$$\frac{\phi_q}{\phi^*} = \begin{cases} \zeta_q(r) - \zeta_q(r_c) + 2\ln r_c, & r \le r_c, \\ 2\ln r, & r_c \le r \le 1 \end{cases} \tag{17}$$

where $\zeta_q(r) \equiv \dfrac{g_q^{1-q}}{q(1-q)\beta} + r^2\dfrac{\lambda}{\beta}$.

Numerical resolution of either Eq. (15) or Eq. (16) taking into account the appropriate constraints, for different values of q, by means of a fourth-order Runge-Kutta method, allowed to obtain the profiles shown in Fig. 1. Let us emphasize at this point that all the $q < 1$ solutions must be, in accordance with the generic form of the generalized canonical equilibrium distributions[4, 5], taken zero for $r \ge r_c$, where r_c is the radial value above which negative solutions become mathematically possible (note that r_c depends on q).

In the particular case $q = 1/2$, Eq. (16) becomes

$$\rho''_{1/2} + \frac{\rho'_{1/2}}{r} = \tfrac{1}{2}(B\rho_{1/2} - A), \tag{18}$$

whose solution is the zeroth order Bessel function, thus reproducing Huang and Driscoll[28] Eq. (3). Fig. 2 shows the profiles corresponding to $q = 1/2$ for different values of the angular momentum L and energy U (we remind that, naturally, every theoretical fitting of the experimental data has to be compatible with the experimental error bars, on L and U, in the present case).

Proposing for Eq. (16), a solution of the form $\rho_q \propto (r_c - r)^\nu$ when $r \to r_c - 0$, we obtain $\nu = \frac{q}{1-q}$, i.e., $\rho_q \propto (r_c - r)^{\frac{q}{1-q}}$. Hence the derivative of the density with respect to r is zero, finite or infinite for $q > 1/2$, $q = 1/2$ and $q < 1/2$, respectively. Therefore, more accurate data for $r \simeq r_c$ could allow the determination, with higher accuracy, of the entropic index q.

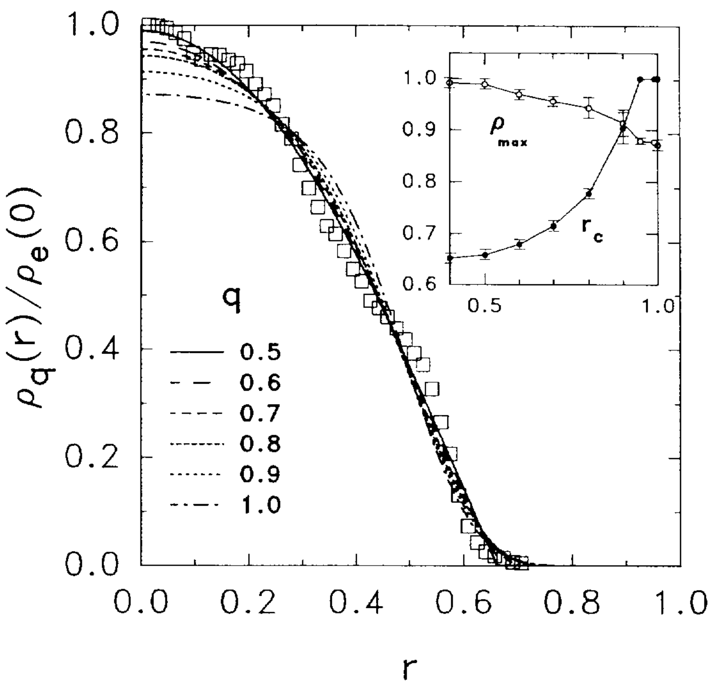

Figure 1: Density profiles of the metaequilibrium state for different values of q, when $L = 0.1392$ and $U = 0.8963$. Square symbols correspond to experimental data. Full lines correspond to theoretical predictions for values of q indicated on the figure. Values are scaled with the experimentally measured density $\rho_e(r = 0)$. **Inset:** r_c and maximal density $\rho_{max} = \rho_q(r = 0)$ vs. parameter q.

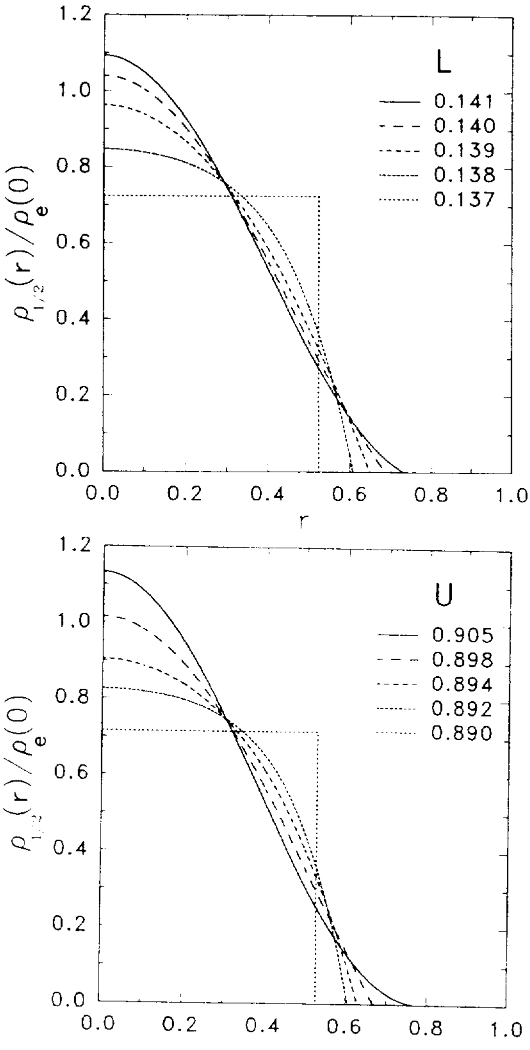

Figure 2: Density profiles of the metaequilibrium state, for $q = 0.5$. (a) for fixed U ($U = 0.896$) and different values of L, (b) for fixed L ($L = 0.139$) and different values of U.

3 Conclusion

The turbulence experiment by Huang and Driscoll[28] constitutes an excellent illustration of the limitations of the Boltzmann-Gibbs thermostatistics. Indeed, their accurate measurements at the metaequilibrium state exclude $q = 1$ and accomodate well with their Restricted Minimum Enstrophy theory shown by Boghosian[13] to be precisely equivalent to the nonextensive $q = 1/2$ theory. The present analysis (Figs. 1 and 2) shows that the experimental data also are compatible with q slightly above 1/2. This possibility is suggested by the fact that, at r_c, the experimental slope of $\rho_q(r)$ seems to vanish.

The present phenomenological calculations have been done within the point vortex picture. More realistic models are of course welcome. It is naturally expected that different models discussed within a given q thermostatistics will yield different profiles, just as did, within the $q = 1$ thermostatistics, Huang and Driscoll's PV and CF models, as well as the four different cosmological models recently considered by Bahcall and Oh[29] (see also [21]) to discuss the peculiar velocity function of galaxy clusters as measured by the COBE satellite. However, it might well happen that they all share one and the same universal behaviour $\rho_q(r) \propto (r - r_c)^{\frac{q}{1-q}}$ in the $r \to r_c - 0$ limit. Finding out the veracity of this conjecture would be very useful, since it would provide an excellent manner for experimentally determining q in turbulent and other complex systems.

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