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Stock index dynamics worldwide: a comparative analysis

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Abstract. We perform a comparative analysis of twenty-four daily stock indices across the world, encompassing developed and emerging markets. We compute, directly from the return empirical time series, the Kramers-Moyal (KM) expansion coefficients that govern the evolution of the probability density function of returns throughout timelags. Our study discloses universal patterns of the KM coefficients, which can be described in terms of a few microscopic parameters. These parameters allow to quantify features such as deviations from Gaussianity or from efficiency, providing a tool to discriminate market dynamics.

PACS. 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.Ey Stochastic processes – 89.65.Gh Economics; econophysics, financial markets, business and management

1 Introduction

Financial markets are archetypal complex systems, consisting of a large number of interacting components subject to a constant flow of information. Some financial observables, such as the fluctuating prices, can be treated as stochastic processes that have been studied in analogy to anomalous diffusion [1], turbulence [2], on-off intermittency [3] and other physical phenomena. As such, the analysis of the dynamics of prices may shed light onto more general processes.

On one hand, the detection of universal patterns amongst markets is important for identifying the fundamental mechanisms that govern market dynamics. On the other, it is also worthwhile to quantify properties of each market, such as efficiency, how it resists to external load, e.g., international speculation, or how long it takes to dissipate the effect of macro-economic shocks.

Comparative studies have been performed before through diverse quantities such as volatility [4–7], Hurst exponents [4,8–11] and other market statistical properties [12,13]. We will show that another useful technique to compare the fluctuating dynamics of stock indices is the evaluation of Kramers-Moyal (KM) coefficients directly from the empirical time series.

From the knowledge of KM coefficients, it is possible to assess quantitatively the deterministic and stochastic laws that govern the dynamics of market prices. This approach has been applied to many economic observables such as stock indices [14–16], exchange rates [17,18] and oil prices [19], reproducing successfully the full evolution of the empirical histograms of returns throughout timelags [14,17].

In the present work, we perform an extensive analysis, by evaluating and comparing the KM coefficients of several daily stock indices worldwide. This study allows to unveil common and distinct features of index return dynamics and, potentially, to rank or segregate markets according to the microscopic parameters that characterize the KM coefficients.

2 Methodology

Previous works have shown that financial data, such as logarithmic returns or exchange rates Δx upon different time lags Δt, can be treated as Markovian stochastic processes, whose empirical PDFs evolve according to the KM expansion:

$$\frac{\partial P(\Delta x,\tau)}{\partial \tau} = \sum_{k\geq 1}\Big[-\frac{\partial}{\partial \Delta x}\Big]^{k} D^{(k)}(\Delta x,\tau)P(\Delta x,\tau). \quad (1)$$

Here, we consider a reverse logarithmic timescale $\tau = \log_2(\Delta t_0/\Delta t)$, where Δt_0 is an arbitrary time lag, set here as 32 (trading) days.

The KM expansion coefficients $D^{(k)}(\Delta x,\tau)$ can be estimated directly from data series, through their statistical definition:

$$D^{(k)}(\Delta x,\tau) = \lim_{\Delta\tau\to 0} \widetilde{D}^{(k)}(\Delta x,\tau,\Delta\tau), \quad (2)$$

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Table 1. World indices and corresponding labels, country and number of trading days in calendar period Jan. 1997–Jul. 2007.

Label	Index	Country	# days
1	AEX	Netherlands	2678
2	ATX	Austria	2597
3	BEL 20	Belgium	2669
4	CAC 40	France	2669
5	DAX	Germany	2662
6	FTSE 100	UK	2652
7	SMI	Switzerland	2647
8	BSE 30	India	2477
9	HSI	Hong Kong	2599
10	JSXC	Indonesia	2428
11	KLSEC	Malaysia	2591
12	KOSPI	South Korea	2490
13	Nikkei 225	Japan	2581
14	STI	Singapore	2643
15	TWI	Taiwan	2462
16	DJIA	USA	2666
17	Nasdaq	USA	2652
18	NYSE	USA	2652
19	S&P 500	USA	2645
20	Ibovespa	Brazil	2596
21	IPC	Mexico	2631
22	Merval	Argentina	2597
23	CMA	Egypt	1964
24	AOX	Australia	2666

with

$$\widetilde{D}^{(k)}(\Delta x, \tau, \Delta\tau) = \frac{1}{k!}\,\frac{M^{(k)}(\Delta x, \tau, \Delta\tau)}{\Delta\tau}, \qquad (3)$$

where $M^{(k)}$ denotes the conditional moments

$$M^{(k)} = \int d\Delta x' (\Delta x' - \Delta x)^k P(\Delta x', \tau + \Delta\tau | \Delta x, \tau). \quad (4)$$

3 Preliminary analysis

We examined twenty-four stock market indices (listed in Tab. 1). This set of indices samples all continents and includes both developed and emerging markets [20]. All data series were collected from the Yahoo's site [21]. We carried out our analysis over a 10-year data window (in the period from January 1997 to July 2007). Time counting was performed over trading days only, skipping weekends and holidays. The measured returns are given in units of the standard deviation σ_0 of the respective data sample at 32-day time-lag.

The stationarity of the return time series was checked by means of two commonly used econometric tests: augmented Dickey-Fuller and Philip-Perron [22]. Although some series present local bursts of volatility, both tests yielded positive diagnostics for stationarity for all examined markets. This means that the data sets allow one to consider, at least to a good degree of approximation, stationary measures. In any case, results can be adopted as average or effective ones for each entire series.

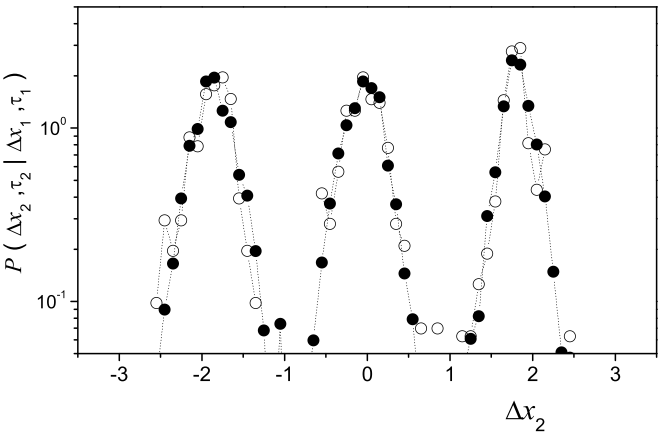

Fig. 1. Cuts through $P(\Delta x_2, \tau_2|\Delta x_1, \tau_1)$ at Δx_1 = -2.0, 0.5 and 3.0 (from left to right) obtained for $\tau_2 = 0.24$, $\tau' = 0.14$ and $\tau_1 = 0.04$, for the Singapore index STI. Histograms were shifted horizontally for better visualization. Filled symbols stand for the directly evaluated conditional PDF and open symbols for the integrated PDF according to equation (5).

In order to investigate the Markovian properties of the stochastic process, we evaluated the Chapman-Kolmogorov (CK) equation [23]:

$$P(\Delta x_2, \tau_2|\Delta x_1, \tau_1) = \int d\Delta x' P(\Delta x_2, \tau_2|\Delta x', \tau')P(\Delta x', \tau'|\Delta x_1, \tau_1) \quad (5)$$

where $\tau_2 > \tau' > \tau_1$. The validity of the CK equation was examined for different triplets of τ_2, τ', τ_1 by comparing the directly evaluated conditional PDF $P(\Delta x_2, \tau_2|\Delta x_1, \tau_1)$ with the integrated PDF calculated from the right-hand side of equation (5). As an illustrative example, in Figure 1, we compare cuts for chosen values of Δx_1 of the two differently computed PDFs for the Singapore index STI. Within statistical errors, the histograms are identical, yielding evidence for the validity of equation (5). Similar results were found for all analyzed markets. The direct check of the CK equation could be also used to estimate the minimum time-lag over which the return series could be approximated by a Markov process [19]. Choosing different Δx_1, we performed the test for several fixed values of τ_1 and decreasing $\tau_2 - \tau'$. No significant deviations were observed until the shortest time resolution of our data [24].

Then we carried on the evaluation of the KM coefficients. We found out that all analyzed datasets furnish, for any $-4 < \tau < 4$ and sufficiently small $\Delta\tau$, a common behavior of the drift coefficient $D^{(1)}$ and the diffusion coefficient $D^{(2)}$ defined in equation (2). In Figure 2a and 2b, we illustrate the typical shapes of coefficients $\widetilde{D}^{(1)}$ and $\widetilde{D}^{(2)}$

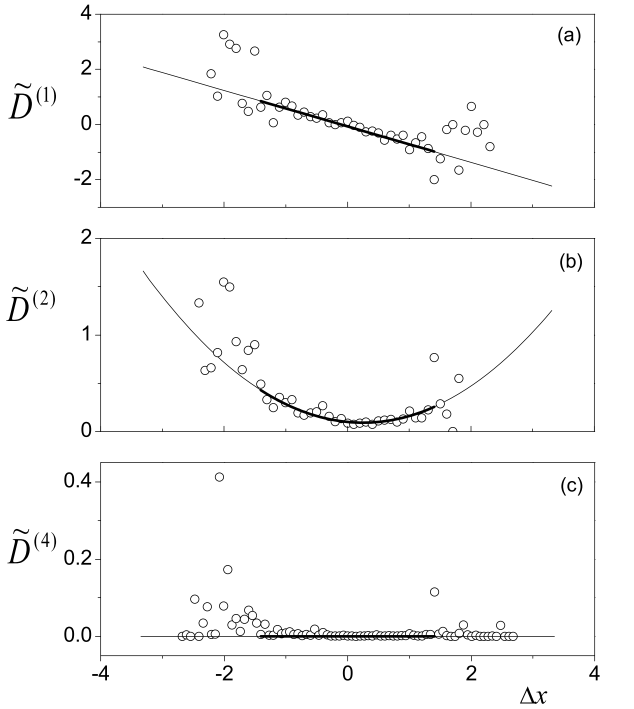

Fig. 2. Coefficients $\widetilde{D}^{(1)}$ (a), $\widetilde{D}^{(2)}$ (b) and $\widetilde{D}^{(4)}$ (c), as a function of Δx, evaluated at $\tau = 1.2$ (weekly timescale) and $\Delta\tau = 0.2$ for German index DAX. Solid lines correspond to fitting curves and thicker pieces to the intervals used for fits.

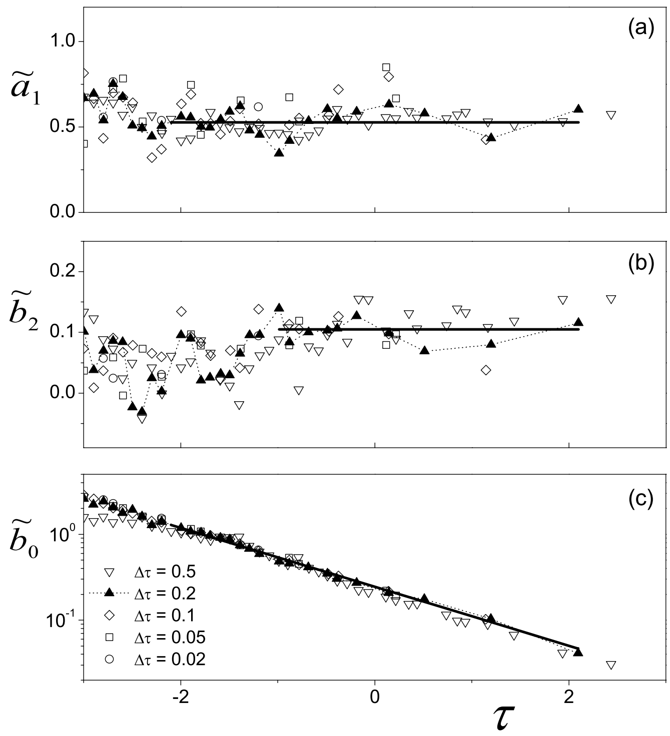

Fig. 3. Dependence of the parameters $\widetilde{a}_1$ (a), $\widetilde{b}_2$ (b) and $\widetilde{b}_0$ (c) on timescale τ, for different values of $\Delta\tau$ indicated in (c), for the Mexican index IPC. Solid lines correspond to the intervals used for calculations, as explained in the text, performed for $\Delta\tau = 0.2$.

as a function of Δx. Thus, in line with results reported in the literature for a restricted set of markets [14–17], we obtained that the Δx-dependence of the drift and diffusion coefficients are well described by linear and quadratic forms, respectively. Namely,

$$\widetilde{D}^{(1)} = -\widetilde{a}_1 \Delta x + \widetilde{a}_0,$$
$$\widetilde{D}^{(2)} = \widetilde{b}_2[\Delta x]^2 + \widetilde{b}_1 \Delta x + \widetilde{b}_0. \quad (6)$$

According to equation (2), the values of $\{\widetilde{a}_i, \widetilde{b}_j\}$, defined in equation (6), determine, in the limit $\Delta\tau \to 0$ for a given τ, the parameters $\{a_i, b_j\}$ that describe the corresponding Δx-dependence of $D^{(1)}$ and $D^{(2)}$.

The fourth order KM coefficient $D^{(4)}$ is consistent with a vanishing value, for all analyzed markets, as suggested in Figure 2c for the German index DAX. Accordingly, equation (1) can be truncated at second order, reducing to a Fokker-Planck equation (FPE). Therefore, this FPE represents a universal evolution equation for the PDFs of returns, fully characterized by a few τ-dependent parameters.

The standard τ-dependence of parameters $\widetilde{a}_i$ and $\widetilde{b}_i$ is exemplified in Figure 3, through the Mexican index IPC, for different values of $\Delta\tau$. We observe that, apart from the fluctuations at the largest timescales ($\tau \leq -2$) due to the poorer statistics, the limiting values ($\Delta\tau \to 0$) of the KM coefficients are fairly achieved for $\Delta\tau \simeq 0.2$, within statistical fluctuations. Then, we perform our comparative analysis by considering the results obtained for $\Delta\tau = 0.2$ as those defining $\{a_i, b_j\}$ at each τ.

As general outcomes for the τ-dependence of the parameters of KM coefficients, we found for all markets the following behaviors, illustrated in Figure 3: (i) Parameter a_1 presents steady behavior within the timescales $-2 \leq \tau \leq 2$ (Fig. 3a). The average of a_1 in that interval is set as the steady value a_1^s. (ii) Parameter b_2 displays an increase with τ from a value close to zero up to a steady level, b_2^s (Fig. 3b). Considering that at $\tau = -1$ all markets have attained a steady level, we estimated b_2^s as the average of b_2 within $-1 \leq \tau \leq 2$. (iii) Parameter b_0 decays exponentially with time as $b_0 = A\,2^{-\gamma\tau}$ (Fig. 3c). We fitted a single exponential law in the interval $-2 \leq \tau \leq 2$, leading to the resulting values A and γ for each market.

Parameter a_0 and b_1 (not shown) have not emerged as significant measures for comparisons. We found negligible parameter a_0 but noticeably negative parameter b_1 for all markets. This result places b_1 as a relevant parameter which governs the significant asymmetries observed in the empirical distributions of returns at the daily-monthly timescales. However, b_1 does not exhibit steady values in these timescales, making its use difficult for comparative purposes. Also, it tends to vanish at large τ, concomitantly with the quenching of asymmetries at intraday scales.

In conclusion, our study shows that the evolution equation for the PDFs of returns reduces to a FPE for all analyzed markets. Each KM coefficient ($D(1)$ and $D(2)$) has the same functional dependence upon Δx for all markets, and the τ-dependence of their parameters also presents

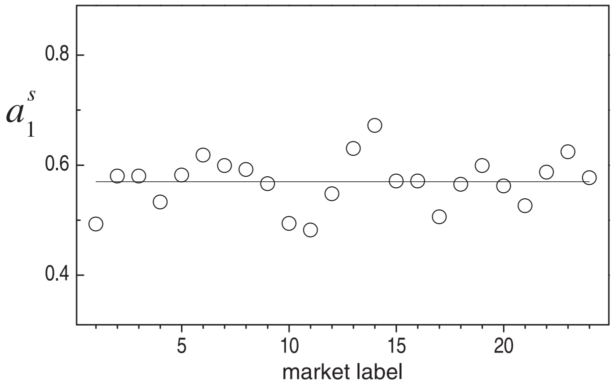

Fig. 4. Measure a_1^s for all the markets listed in Table 1, according to their labels. The error of a_1^s is about 10%. The full line corresponds to the average value 0.57 ± 0.05.

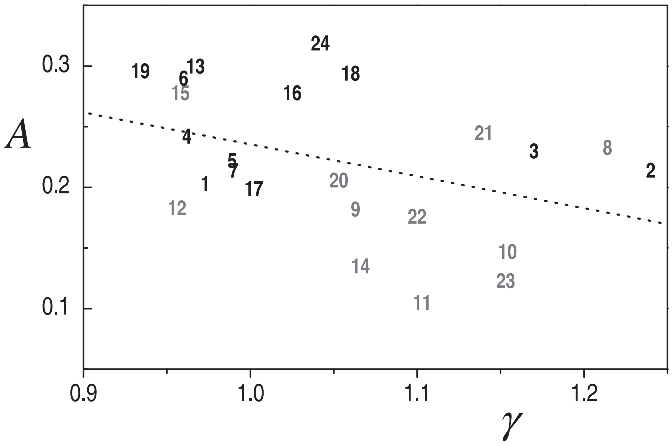

Fig. 5. Market representation in the plane of parameters (γ,A), according to the numerical labels in Table 1. Symbols are darker for developed markets. Errors are about 5%. The dotted line is a linear regression as guide to the eye.

universal features for the monthly/weekly timescale considered. Our preliminary analysis has pointed to the following set of relevant quantities, computed as defined above: the steady values a_1^s and b_2^s, and parameters A and γ that characterize the τ-dependence of b_0.

4 Comparative analysis

Along this section, we will use the numeric labels shown in Table 1 to identify markets. After having detected dynamical properties qualitatively shared by all those markets, now, we proceed to perform quantitative comparisons of the measures that came out from the preliminary analysis.

The measure a_1^s estimates the coefficient of the market restoring force against price fluctuations. Its value for each market is presented in Figure 4. We found a universal behavior across markets, with average value 0.57 and standard deviation 0.05 (smaller than the error of a_1^s). Although this outcome discards a_1^s as a classifying measure, it is a relevant result since it manifests that the deterministic laws that rule market dynamics are universal not only qualitatively but also quantitatively, at the analyzed timescales.

In Figure 5 we represent the markets (with darker symbols for developed ones) in the plane (γ, A) of the two free parameters of the additive noise component b_0. While A describes the amplitude of b_0 at $\tau = 0$, γ is the inverse decay time. From this figure, one sees that A presents a wide dispersion across markets (a factor about three between extreme values). Figure 5 also shows that developed markets, except for those of Austria and Belgium (labels 2 and 3, respectively), cluster in the region of large A and small γ. This finding suggests that this set of parameters has potential ability to classify markets. Note that large A and small γ imply that additive noise components of index fluctuations are large and persistent over timescales.

As a matter fact, the state-independent contribution to the diffusion coefficient scales with the timelag Δt as $b_0 = A2^{-\gamma\tau} \propto (\Delta t)^\gamma$. Then, deviations from linearity ($\gamma \neq 1$) yield anomalous diffusion, suggesting that γ may play a similar role to that of the Hurst exponent H, which is considered a measure of efficiency. Market indices have been ranked in the literature according to their Hurst exponents (usually associated to the scaling of the variance) by means of different approaches [8–11]. It is worth to mention that although some works have also reported non-stationary Hurst exponents [10,11,25], only global or average ones furnish a tool for comparisons.

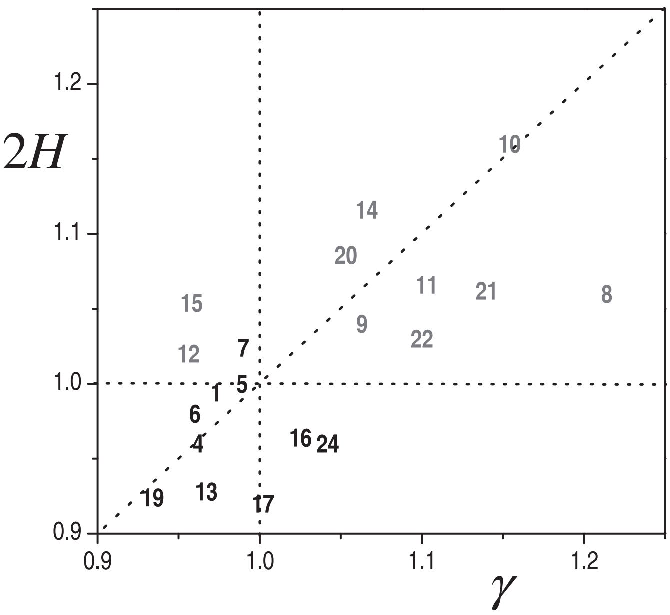

Fig. 6. Market representation according to the measures $2H$ [8] and γ (this work). Symbols are as in Figure 5. The error of γ is about 5%. Dotted borderlines ($\gamma = 1$ and $2H = 1$) are drawn for comparisons.

Then, we carried on a comparison between both exponents. In Figure 6, we display γ, measured in this work, and H, reported in the literature. We considered reference [8], due to the many common markets in both samples, including developed and emerging ones, as well as to the matching of the analyzed calendar periods. Figure 6 shows clearly a positive correlation between both parameters. Moreover, there is tendency for mature markets to rank as the most efficient ones, with values of γ (and $2H$) smaller than those for emerging markets. The borderlines of efficiency ($\gamma = 1$ and $2H = 1$) also show up the good accord between the segregations yielded by each measure, qualifying γ as a meaningful classifying parameter for market efficiency.

The values of b_2^s, which describes the amplitude of the multiplicative random component in the price fluctuation mechanisms, present a wide relative variation amongst markets (about a factor three between the smallest and largest detected values). However, the mere display of the values of b_2^s does not manifest a clear economic order or classification amongst markets. In fact, a more meaningful quantity to characterize market dynamics is the relative strength of multiplicative and additive noise components. It will be discussed below in connection with the tails of the PDFs of returns.

Assuming, in accord with our observations, exponential τ-dependence of b_0, steady values for a_1 and b_2, as well as negligible a_0 and negligible asymptotic value of b_1, the resulting FPE admits an asymptotic solution of the form [14]:

$$P(\Delta x, \tau) = P_0/(b_0(\tau) + b_2[\Delta x]^2)^{(\mu+1)/2} , \qquad (7)$$

where P_0 is a normalization constant. The variance computed with PDF (7) is $\sigma^2(\tau) = A2^{-\gamma\tau}/(b_2[\mu-2])$. Because of the scaling chosen for Δx, the variance is unitary at $\tau = 0$, then one has

$$\mu = 2 - A/b_2 . \qquad (8)$$

For consistency of the solution, the KM parameters can not be independent but must be related through $A = a_1 - b_2 - \gamma \ln 2/2$. This relation holds only approximately since the asymptotic limit has not been attained at the analyzed timescales. Similarly, the theoretical value of μ, given by equation (8), determines the exponent of the tails of the PDF only asymptotically for large τ (small time-lags). Still, this limit is reached only if the exponential τ-dependence or b_0, together with levels a_1 and b_2 persisted for many hierarchies of timescales.

In order to exploit possible links between the microscopic measures and the observational property of the PDFs tails, we also obtained empirical values of μ, μ_{emp}, at the daily scale ($\tau = 5$), the largest timescale of our data. They were computed from a least-squares fit of the standardized form of equation (7): $P(x) = \frac{\Gamma((\mu+1)/2)}{\Gamma(\mu/2)\sqrt{\pi(\mu-2)}}(1 + \frac{x^2}{\mu-2})^{-(\mu+1)/2}$ (with only one fitting parameter, μ), as illustrated in Figure 7 for Argentinian index Merval and USA index Dow Jones.

The markets are represented in the plane of exponent μ_{emp} and ratio b_2^s/A in Figure 8. One observes a remarkable correlation between both quantities that follow approximately the theoretical relation given by equation (8). This suggests that a relevant content of information has been catched by the multiplicative/additive ratio.

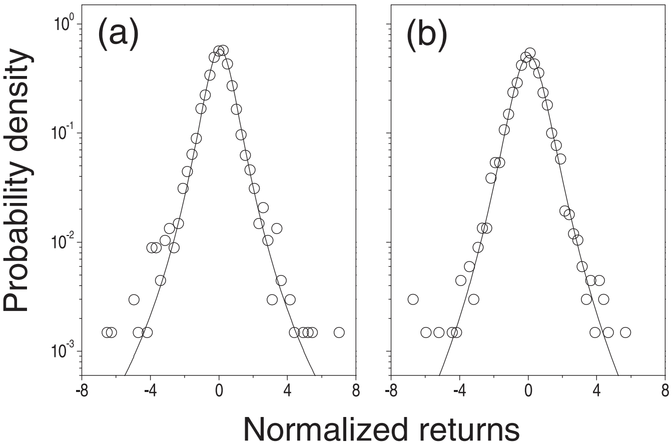

Fig. 7. Standardized PDFs of daily returns for the Argentinian index Merval (a) and USA index Dow Jones (b). Symbols correspond to empirical histograms, solid lines to least squares fits using the standardized form of equation (7).

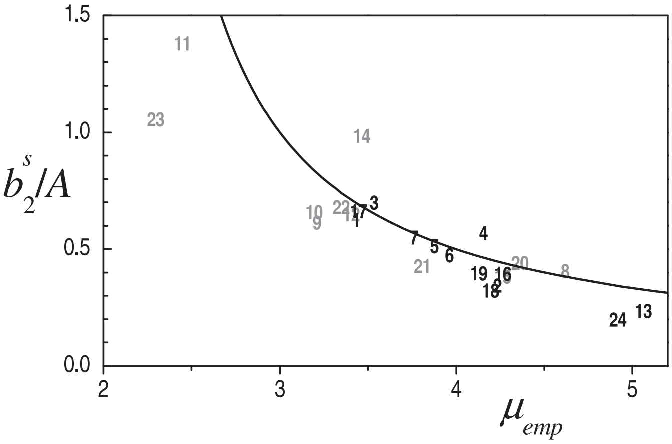

Fig. 8. Markets represented in the plane ($\mu_{emp}, b_2^s/A$). Symbols are as in Figure 5. Developed markets are darker. The dotted line is the locus of the points given by equation (8).

In additive-multiplicative processes, the role of the later noise for yielding power-law tails is well known [26,27]. Accordingly, larger ratio b_2^s/A leads to smaller μ_{emp}, supporting stronger deviations from Gaussianity, which is recovered in the limit $\mu \to \infty$. One also observes the trend followed by developed markets, as well as by large emerging ones, to have smaller ratio b_2^s/A. This result indicates that the composite measure b_2^s/A has a potential classifying role. Moreover, independently of γ, it represents an important perspective for examining efficiency: the departure from Gaussianity.

5 Summary

We have analyzed the KM coefficients that govern the price dynamics at macro timescales for a representative set of stock markets worldwide. As a general result, we found that the evolution equation for the PDFs of returns can be properly represented by a universal FPE, fully characterized by a few τ-dependent parameters. Specifically, our investigation reveals that drift and diffusion coefficients have the same functional dependence upon Δx, and that the τ-dependence of their parameters also presents universal characteristics at the monthly/weekly timescales. Parameters endowed with significant information on market dynamics were analyzed to accomplish possible classifying measures.

The steady levels a_1^s, describing the coefficient of the restoring force drawn from $D^{(1)}$, do not present significant differences across markets (Fig. 4). This result discards a_1^s as a potential ranking measure. However, it expresses an important finding: the deterministic rules that constrain the dynamics of most markets at the monthly/weekly time horizons act with similar intensity. Therefore, relaxation mechanisms are universal both qualitative and quantitatively, despite the variety of flux of information, stock liquidity, and risk aversion from market to market.

Otherwise, our study has shown that, although qualitatively similar, both additive and multiplicative stochastic components, drawn from $D^{(2)}$, are quantitatively distinct, constituting important parameters to distinguish amongst markets. In particular, the additive noise component b_0 decays exponentially (with the reverse logarithmic timescale τ) with exponent γ, which, similarly to the Hurst exponent, arises as a classifying measure of market efficiency (Fig. 6). Moreover, A, the characteristic amplitude of b_0 at monthly timescales, presents a wide dispersion across markets, allowing segregation of mature and emerging markets over the (γ, A) plane (Fig. 5). The multiplicative noise component of the monthly/weekly timeframes, b_2^s, also exhibits a wide relative variation. Although individually it does not allow a clear link with the economic stage of the markets, we found that the ratio b_2^s/A provides a probe of the deviation of the empirical histograms from normality at the daily timescale (Fig. 8). The above results are also grounded on analytical calculations based on the scaled PDF solution of the asymptotic FPE, which relates microscopic measures to macroscopic observable quantities, namely, the variance and the tail exponent of the PDFs.

In sum, the present analysis have disclosed universal features of stock index stochastic dynamics. Otherwise, it allows to distinguish markets by means of inner parameters containing significant information on market dynamics.

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