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Curl forces and the nonlinear Fokker-Planck equation

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Nonlinear Fokker-Planck equations endowed with curl drift forces are investigated. The conditions under which these evolution equations admit stationary solutions, which are q exponentials of an appropriate potential function, are determined. It is proved that when these stationary solutions exist, the nonlinear Fokker-Planck equations satisfy an H theorem in terms of a free-energy-like quantity involving the S_q entropy. A particular two-dimensional model admitting analytical, time-dependent q-Gaussian solutions is discussed in detail. This model describes a system of particles with short-range interactions, performing overdamped motion under drag effects due to a rotating resisting medium. It is related to models that have been recently applied to the study of type-II superconductors. The relevance of the present developments to the study of complex systems in physics, astronomy, and biology is discussed.

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I. INTRODUCTION

The nonlinear Fokker-Planck equation [1] constitutes a powerful tool for the study of diverse phenomena in complex systems [2–8], with applications including (among many others) type-II superconductors [9], granular media [10], and self-gravitating systems [11,12]. It governs the behavior of a time-dependent density $F(\boldsymbol{x},t)$, where $\boldsymbol{x} \in \Re^N$ designates a location in an N-dimensional configuration space. The evolution of F is determined by two terms: a nonlinear diffusion [13,14] term and a linear drift term (more general equations with nonlinear drift terms have also been proposed, but we are not going to consider them in the present work). In several of the above-mentioned applications, the density F is a real physical density (as opposed to a statistical ensemble probability density) describing the evolving distribution of a set of interacting particles executing overdamped motion in the relevant configuration space [8,15]. In these kinds of scenarios, the nonlinear diffusion term constitutes an effective description of the interaction between the particles, while the drift term describes the effects of other external forces acting upon them. The nonlinear Fokker-Planck equations recently addressed in the literature exhibit several interesting and physically relevant properties. They obey an H theorem in terms of a free-energy-like quantity [16]. In some important cases, the nonlinear Fokker-Planck equations admit exact analytical solutions of the q-Gaussian form that can be interpreted as maximum entropy (q-maxent) densities obtainable from the optimization under appropriate constraints of the nonadditive power-law entropic functionals S_q [17,18]. Indeed, there is a deep connection between the nonlinear Fokker-Planck dynamics and the generalized thermostatistics based on the S_q entropies. Although this connection was first pointed out more than 20 years ago [19], its full physical implications have been systematically explored only in recent years (see, for instance, [2,15,20–22] and references therein). A remarkable example of this trend is given by experimental work on granular media [10] that verifies, within a 2% error and for a wide experimental range, a scale relation predicted earlier on the basis of the theoretical analysis of q-Gaussian solutions of the nonlinear Fokker-Planck equation [23]. The particular case of the nonlinear Fokker-Planck equation with vanishing drift corresponds to the porous media equation.

Virtually all the literature on the nonlinear Fokker-Planck equation and its applications deals with Fokker-Planck equations in which the drift forces $\boldsymbol{K}$ can be derived from a potential function $V(\boldsymbol{x})$, leading to stationary densities that are q exponentials of the potential V. In the present paper we consider more general scenarios where the drift force $\boldsymbol{K}$ has, besides a component given by minus the gradient of a potential V, a term $\tilde{\boldsymbol{K}}$ that does not come from a potential. In two or three space dimensions, this situation corresponds to having forces exhibiting a nonvanishing rotational or curl, which are usually referred to as curl forces [24]. The incorporation of curl forces enriches the dynamical features of the nonlinear Fokker-Planck equations, enabling it to describe a wider set of phenomena.

Curl forces, although not dynamically fundamental [25], are nevertheless relevant as useful effective descriptions of diverse physical problems, for example, the nonconservative force fields generated by optical tweezers [26]. Dynamical systems with curl forces have interesting properties that are not yet fully understood and are the subject of current research [25,27]. In the present work we investigate the behavior of nonlinear Fokker-Planck equations under the presence of curl forces. We determine the conditions under which these evolution equations admit stationary solutions of the

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q-maxent form and satisfy an H theorem. We also discuss in detail a two-dimensional example admitting analytical time-dependent solutions that describes a set of interacting particles undergoing overdamped motion under the drag effect arising from a uniformly rotating medium.

II. NONLINEAR FOKKER-PLANCK EQUATION

In the present work we consider nonlinear Fokker-Planck (NLFP) equations of the form

$$\frac{\partial F}{\partial t} = D\nabla^2[F^{2-q}] - \nabla \cdot [F\boldsymbol{K}], \tag{1}$$

where $F(\boldsymbol{x},t)$ is a time-dependent density, D is a diffusion constant, $\boldsymbol{K}(\boldsymbol{x})$ is a drift force, and q is a real parameter characterizing the (power-law) nonlinearity appearing in the diffusion term. The density F is a dimensionless quantity of the form $F = \rho(\boldsymbol{x},t)/\rho_0$, where ρ has dimensions of inverse volume and ρ_0 is a constant with the same dimensions as ρ. Therefore, the dimensional density $\rho(\boldsymbol{x},t)$ obeys the evolution equation $\partial[\rho/\rho_0]/\partial t = D\nabla^2[(\rho/\rho_0)^{2-q}] - \nabla \cdot [(\rho/\rho_0)\boldsymbol{K}]$.

As already mentioned, in the most frequently studied case of Eq. (1), the drift force $\boldsymbol{K}$ is assumed to arise from a potential function $V(\boldsymbol{x})$,

$$\boldsymbol{K} = -\nabla V. \tag{2}$$

The stationary solutions of the NLFP equation then satisfy

$$\nabla \cdot [D\nabla(F^{2-q}) + F(\nabla V)] = 0. \tag{3}$$

Let us consider the q-statistical ansatz [18]

$$\begin{aligned} F_q &= A \exp_q[-\beta V(\boldsymbol{x})] \\ &= A[1-(1-q)\beta V(\boldsymbol{x})]_+^{1/(1-q)}, \end{aligned} \tag{4}$$

where A and β are constants to be determined and the function $\exp_q(z) = [1+(1-q)z]_+^{1/(1-q)}$, usually referred to as the q-exponential function, vanishes whenever $1+(1-q)z \leqslant 0$. One finds that the ansatz given by (4) complies with the equation

$$D\nabla(F^{2-q}) + F(\nabla V) = 0, \tag{5}$$

if

$$(2-q)\beta D = A^{q-1}. \tag{6}$$

It therefore satisfies also Eq. (3) and constitutes a stationary solution of the NLFP equation. In summary, the q-exponential ansatz (4) is a stationary solution of the NLFP equation if the drift force $\boldsymbol{K}$ is derived from a potential and A and β satisfy the relation (6).

We will assume that the stationary distribution F_q has a finite norm, that is, $\int F_q\, d^N\boldsymbol{x} = I < \infty$. The specific conditions required for F_q to have a finite norm (such as the allowed range of q values) cannot be stated in general because they depend on the particular form of the potential function $V(\boldsymbol{x})$. Since in many applications the solution of the NLFP equation is interpreted as a physical density (as opposed to a probability density), we assume a finite norm, but not necessarily normalization to unity. Note that, due to the nonlinear character of the Fokker-Planck equation under consideration, the multiplicative factor A appearing in the solution (4) cannot be chosen arbitrarily (as would be the case for the linear Fokker-Planck equation). The parameter A depends on the value of β through relation (6), implying that a stationary solution normalized to one corresponds to a particular β value, which in turn depends on the value of q.

The stationary density F_q can be regarded as a q-maxent distribution, because it maximizes the nonadditive q-entropic functional

$$S_q[F] = \frac{1}{q-1}\int (F - F^q)d^N\boldsymbol{x}, \tag{7}$$

under the constraints corresponding to the norm and the mean value of the potential V [18,19]. In the limit $q \to 1$, the standard linear Fokker-Planck equation

$$\frac{\partial F}{\partial t} = D\nabla^2 F - \nabla \cdot [F\boldsymbol{K}] \tag{8}$$

is recovered. In this limit, the q-maxent stationary density (4) reduces to the exponential, Boltzmann-Gibbs-like density

$$F_{\mathrm{BG}} = \frac{1}{Z}\exp\left[-\frac{1}{D}V(\boldsymbol{x})\right], \tag{9}$$

with the condition (6) becoming $\beta D = 1$, independent of the normalization constant A. The density F_{BG} is normalized to one provided that $Z = \int \exp[-\frac{1}{D}V(\boldsymbol{x})]d^N\boldsymbol{x}$. The density F_{BG} optimizes the Boltzmann-Gibbs entropy $S_{\mathrm{BG}} = -\int F\ln F d^N\boldsymbol{x}$, under the constraints of normalization and the mean value $\langle V\rangle$ of the potential V (consistently with the fact that, in the limit $q \to 1$, the entropic functional S_q reduces to the standard Boltzmann-Gibbs entropy).

Note that a dynamical system with a phase space flux of the form (2) (that is, of a gradient form) evolves always downhill on the potential energy landscape so as to minimize the potential energy function $V(\boldsymbol{x})$. The components $\{K_i, i = 1,\ldots,N\}$ of such a field satisfy

$$\frac{\partial K_i}{\partial x_j} = \frac{\partial K_j}{\partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j}. \tag{10}$$

In two or three dimensions, as is well known, fields $\boldsymbol{K}$ that do not have the gradient form are fields of nonvanishing curl, i.e., $\boldsymbol{K} \neq -\nabla V \iff \nabla \times \boldsymbol{K} \neq 0$.

III. NONLINEAR FOKKER-PLANCK EQUATION WITH CURL DRIFT FORCES: STATIONARY SOLUTIONS

Now we consider NLFP equations endowed with drift forces having two terms, one exhibiting the gradient form and the other not arising from the gradient of a potential. That is, we consider drift fields of the form

$$\boldsymbol{K} = \boldsymbol{G} + \tilde{\boldsymbol{K}}, \tag{11}$$

where the force $\boldsymbol{G}$ is equal to minus the gradient of some potential function $V(\boldsymbol{x})$, while the component $\tilde{\boldsymbol{K}}$ does not come from a potential (that is, $\partial\tilde{K}_i/\partial x_j \neq \partial\tilde{K}_j/\partial x_i$). Our aim is to determine under which conditions a density proportional to the q exponential of the potential V still provides a stationary solution of the NLFP equation, preserving thus the link between this equation and the generalized nonextensive thermostatistics. Substituting the above drift force $\boldsymbol{K}$ and q-exponential density F_q (4) into the stationary NLFP equation (3), one obtains

$$D\nabla^2\big[F_q^{2-q}\big] + \nabla\cdot[F_q(\nabla V)] - \nabla\cdot[F_q\tilde{\boldsymbol{K}}] = 0. \qquad (12)$$

It can be verified that, if A and β satisfy (6), the sum of the first two terms in the above equation vanishes, since F_q is a stationary solution of the NLFP equation (3) when the drift field $\boldsymbol{K}$ consists solely of the gradient field $\boldsymbol{G}$.

In order for F_q to comply also with the full NLFP equation (12), including the drift contribution associated with the nongradient field $\tilde{\boldsymbol{K}}$, it is then necessary that

$$\nabla\cdot[F_q\tilde{\boldsymbol{K}}] = 0. \qquad (13)$$

If the above relation is satisfied, the density F_q constitutes a stationary solution of the full NLFP equation, corresponding to the complete drift force $\boldsymbol{K} = -(\nabla V) + \tilde{\boldsymbol{K}}$. To have the q-maxent stationary solution, one therefore requires

$$\nabla\cdot\{\tilde{\boldsymbol{K}}A[1-(1-q)\beta V]^{1/(1-q)}\} = 0, \qquad (14)$$

which in turn leads to the following relation between the nongradient drift component $\tilde{\boldsymbol{K}}$ and the potential function $V(\boldsymbol{x})$:

$$[1-(1-q)\beta V](\nabla\cdot\tilde{\boldsymbol{K}}) - \beta(\tilde{\boldsymbol{K}}\cdot\nabla V) = 0. \qquad (15)$$

This is a consistency relation that the potential function V, the nongradient force field $\tilde{\boldsymbol{K}}$, the Lagrange multiplier β, and the entropic parameter q have to satisfy in order that the nonlinear Fokker-Planck equation admits the q-maxent stationary solution (4). The general (q,β)-dependent equation (15) constitutes a rather complicated relation between the nongradient field $\tilde{\boldsymbol{K}}$ and the potential function V, which is difficult to characterize. Moreover, this relation depends explicitly on the value of β (besides depending, of course, on the value of the q entropic parameter). This means that, for given forms of $\tilde{\boldsymbol{K}}(\boldsymbol{x})$ and $V(\boldsymbol{x})$ and a given q value, one may have stationary solutions of the q-maxent form (4), only for particular values of $\beta = \beta(q)$, which will in general be functions of q, the β dependence and the q dependence being thus highly intertwined. We will not pursue further an analysis of these kinds of scenarios and will focus instead on the case where (for given q values) there are solutions for a continuous range of β values. In this case, we will see that condition (15) decouples into two separate conditions, each of them independent of both q and β. Having a continuous range of allowed β values has the important advantage of giving us the freedom to choose solutions with different normalizations [note that the constant A in the stationary solution (4) is a function of β]. In particular, one can choose solutions normalized to one. Regarding this last point, it is worth mentioning here that the q dependence of β reappears when one considers only solutions normalized to unity [see the discussion after Eq. (51)].

It follows from relation (15) that, in order for the NLFP equation to admit the β-parametrized family of stationary solutions (4), with a continuous allowed range of β values, two conditions have to be fulfilled. On the one hand, the nongradient component of the drift $\tilde{\boldsymbol{K}}$ has to be a divergence-free vector field

$$\nabla\cdot\tilde{\boldsymbol{K}} = 0. \qquad (16)$$

On the other hand, $\tilde{\boldsymbol{K}}$ has to be everywhere orthogonal to the gradient of the potential

$$\tilde{\boldsymbol{K}}\cdot(\nabla V) = 0. \qquad (17)$$

Notice that conditions (16) and (17) are not only sufficient but also necessary conditions for the ansatz (4) to be a stationary solution of the NLFP equation (1) for a continuous range of β values. Indeed, if (4) solves (1) for such a set of β values, the left-hand side of (15), which is an inhomogeneous linear function of β, has to vanish for an interval of values of β. This clearly implies that both the independent term and the coefficient of the β-linear term have to vanish individually, leading in turn to conditions (16) and (17). It is interesting that, as already mentioned, these conditions do not explicitly depend on the value of the q parameter, constituting therefore a q-invariant structure. This q invariance is remarkable since it incorporates both the $q < 1$ and $q > 1$ regimes, which in specific applications usually correspond to quite different dynamics. The stationary solution guaranteed by these conditions is a physical solution when it is normalizable (otherwise, it is not physical, although still formally a solution of the NLFP equation). The normalizability of the stationary solution depends on the particular shape of the potential V and on the value of q and, as already mentioned, can only be studied on a case by case basis. The general properties of the NLFP equations with curl forces analyzed in Secs. III–VI are valid for general q values provided this normalizability condition is satisfied.

In two or three space dimensions, the decomposition $\boldsymbol{K} = \boldsymbol{G} + \tilde{\boldsymbol{K}}$, with $\boldsymbol{G} = -\nabla V$ and $\nabla\cdot\tilde{\boldsymbol{K}} = 0$, resembles the decomposition of a vector field into a curl-free (irrotational) component and a solenoidal (divergence-free) component arising from the celebrated Helmholtz theorem [28]. We are not, however, imposing the boundary conditions on the fields $\boldsymbol{K}$, $\boldsymbol{G}$, and $\tilde{\boldsymbol{K}}$, which are usually considered in connection with the Helmholtz decomposition. Furthermore, we require the point to point orthogonality of the irrotational and the divergence-free components of $\boldsymbol{K}$, which is not a condition usually considered in connection with the Helmholtz decomposition.

It is interesting that the Helmholtz-like decomposition (11), with orthogonal irrotational and divergence-free parts $\boldsymbol{G}\cdot\tilde{\boldsymbol{K}} = 0$, arises naturally in some circumstances. For instance, the most general rotationally invariant vector field in two dimensions has precisely this form. Indeed, such vector fields are of the form

$$\boldsymbol{G} = -g(r)\boldsymbol{e}_r,$$
$$\tilde{\boldsymbol{K}} = l(r)\boldsymbol{e}_\theta, \qquad (18)$$

where $g(r)$ and $l(r)$ are functions of the radial coordinate $r = (x^2+y^2)^{1/2}$ and $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ respectively denote the radial and tangential unit vectors. It is clear that the field $\boldsymbol{G}$ in (18) is of the form $-\nabla V(r)$, with $V(r) = \int^{r'} g(r')dr'$, and that the field $\tilde{\boldsymbol{K}}$ satisfies $\nabla\cdot\tilde{\boldsymbol{K}} = 0$ and $\boldsymbol{G}\cdot\tilde{\boldsymbol{K}} = 0$.

Summing up, we have thus determined that the NLFP equation (1) having a nonpotential drift force of the form (11) admits, for a continuous range of values of the parameter β, the family of q-maxent stationary solutions (4) if and only if the relations (16) and (17) are satisfied.

IV. THE H THEOREM

We are now going to explore the possibility of formulating an H theorem for the nonlinear Fokker-Planck equations, endowed with a drift term involving a nonvanishing-curl force $\tilde{\boldsymbol{K}}$, not derivable from the potential function V. Let us first consider the time derivative of the power-law entropic functional S_{q^*}, with $q^* = 2 - q$. This is a reasonable choice, because q^* is precisely the exponent that appears inside the Laplacian term in the NLFP equation (1). The duality $q \to 2 - q$ appears frequently in the q-generalized thermostatistical formalism [18]. We have

$$\frac{dS_{q^*}}{dt} = \frac{q^*}{1-q^*}\int F^{q^*-1}\frac{\partial F}{\partial t}d^N\boldsymbol{x} = Dq^{*2}\int F^{2q^*-3}|\nabla F|^2 d^N\boldsymbol{x} + q^*\int F^{q^*-1}(\nabla F)\cdot(\nabla V)d^N\boldsymbol{x} + \int F^{q^*}(\nabla\cdot\tilde{\boldsymbol{K}})d^N\boldsymbol{x}, \tag{19}$$

where we have used the norm preservation, i.e., $\frac{d}{dt}\int F d^N\boldsymbol{x} = \int \frac{\partial F}{\partial t}d^N\boldsymbol{x} = 0$. It is clear that the first term in the final expression in (19) is positive definite. However, the second and third terms do not have a definite sign. Consequently, the time derivative of S_{q^*} does not have a definite sign and the entropic form S_{q^*} does not itself verify an H theorem. The last two terms in the expression for $\frac{dS_{q^*}}{dt}$, describing the contribution of the drift term to the change in the entropy, suggest that a linear combination of S_{q^*} and the mean value of the potential function V may comply with an H theorem. The time derivative of $\langle V\rangle = \int FVd^N\boldsymbol{x}$ is

$$\frac{d\langle V\rangle}{dt} = \int V\frac{\partial F}{\partial t}d^N\boldsymbol{x} = -q^*D\int F^{q^*-1}(\nabla F)\cdot(\nabla V)d^N\boldsymbol{x} - \int F|\nabla V|^2 d^N\boldsymbol{x} + \int F(\nabla V)\cdot\tilde{\boldsymbol{K}}d^N\boldsymbol{x}. \tag{20}$$

Combining now Eqs. (19) and (20) one obtains, after some algebra,

$$\frac{d}{dt}(DS_{q^*} - \langle V\rangle) = \int F|q^*DF^{q^*-2}(\nabla F) + \nabla V|^2 d^N\boldsymbol{x} + \int F^{q^*}(\nabla\cdot\tilde{\boldsymbol{K}})d^N\boldsymbol{x} + \int F(\nabla V)\cdot\tilde{\boldsymbol{K}}d^N\boldsymbol{x}. \tag{21}$$

If the curl component $\tilde{\boldsymbol{K}}$ of the drift force complies with the requirements given by Eqs. (16) and (17), which are necessary and sufficient for the nonlinear Fokker-Planck equation to have the family of q-maxent stationary solutions (4), it follows from (21) that the nonlinear Fokker-Planck equation satisfies the H theorem

$$\frac{d}{dt}(DS_{q^*} - \langle V\rangle) = \int F|q^*DF^{q^*-2}(\nabla F) + \nabla V|^2 d^N\boldsymbol{x} = \langle|q^*DF^{q^*-2}(\nabla F) + \nabla V|^2\rangle \geqslant 0. \tag{22}$$

It is worth stressing that the conditions (16) and (17) for having stationary q-maxent solutions are essentially the same as those for having an H theorem.

There is an interesting consequence of the H theorem in relation to the uniqueness of the decomposition (11) of the total drift force $\boldsymbol{K}$ into a gradient component $\boldsymbol{G} = -\nabla V$ and an (orthogonal) divergence-free component $\tilde{\boldsymbol{K}}$. Let us assume that that total drift force can be decomposed in this fashion in two different ways, $\boldsymbol{K} = -\nabla V_1 + \tilde{\boldsymbol{K}}_1 = -\nabla V_2 + \tilde{\boldsymbol{K}}_2$. If the nonlinear Fokker-Planck equation admits a stationary solution (of finite norm) $F_{\rm stat}$, it follows from the H theorem (22) that

$$\nabla V_1 = \nabla V_2 = -q^*DF_{\rm stat}^{q^*-2}(\nabla F_{\rm stat}), \tag{23}$$

which in turn implies also that $\tilde{\boldsymbol{K}}_1 = \tilde{\boldsymbol{K}}_2$. Consequently, if the nonlinear Fokker-Planck equation admits a stationary solution, the decomposition of the total drift force into the sum of a gradient term and a divergence-free term is unique.

V. QUADRATIC POTENTIAL AND LINEAR DRIFT

We now consider in detail the case of a quadratic potential V and a linear drift $\tilde{\boldsymbol{K}}$. We will see that, in this case, the conditions (16) and (17) are required even to have a stationary solution of the q-exponential form (4) for one single value of the parameter β. We assume a potential and a drift field, respectively, of the forms

$$V(\boldsymbol{x}) = \sum_{ij}(a_{ij}x_ix_j) + \sum_i(b_ix_i), \tag{24}$$

$$\tilde{K}_i(\boldsymbol{x}) = \sum_j(c_{ij}x_j) + d_i, \tag{25}$$

where $\tilde{K}_i(\boldsymbol{x})$ is the ith component of the drift field $\tilde{\boldsymbol{K}}(\boldsymbol{x})$ and a_{ij}, c_{ij}, b_i, and d_i are constant coefficients. We can assume $a_{ij} = a_{ji}$, although the c_{ij} are not necessarily symmetric. Equation (15) leads to a set of constraints on these coefficients, thus defining $V(\boldsymbol{x})$ and $\tilde{\boldsymbol{K}}(\boldsymbol{x})$. If we substitute Eqs. (24) and (25) in (15), we obtain

$$\left\{1-(1-q)\beta\left[\sum_{ij}(a_{ij}x_ix_j)+\sum_i(b_ix_i)\right]\right\}\left(\sum_k c_{kk}\right) - \beta\sum_k\left[\sum_i(c_{ki}x_i)+d_k\right]\left[\sum_j((a_{kj}+a_{jk})x_j) + b_k\right] = 0. \tag{26}$$

Equation (26) is a second degree polynomial in the x_i that is equal to zero. Since this equality should hold for any value of $\boldsymbol{x}$, the coefficients of the different powers of the x_i should each be equal to zero. Therefore, by separately equating to zero the independent zeroth-, first-, and second-order terms on the left-hand side of (26), one obtains

$$\sum_k(c_{kk} - \beta d_kb_k) = 0, \tag{27a}$$

$$\sum_k[(1-q)c_{kk}b_i + c_{ki}b_k + (a_{ki}+a_{ik})d_k] = 0\,\forall i, \tag{27b}$$

$$\sum_k[(1-q)c_{kk}(a_{ij}+a_{ji}) + c_{ki}(a_{kj}+a_{jk}) + c_{kj}(a_{ki}+a_{ik})] = 0\,\forall i,j. \tag{27c}$$

With symmetric a_{ij}, we now assume

$$\det|a_{ij}| \neq 0. \tag{28}$$

This assumption is also necessary if $V(\boldsymbol{x})$ should represent a confining potential, leading to a normalizable stationary state of the nonlinear Fokker-Planck equation.

If we introduce an appropriate shift in the x_i coordinates, it is possible to work using a potential $V(\boldsymbol{x})$ (24) with no linear terms. We thus define

$$\overline{x}_i = x_i - r_i, \tag{29}$$

so that the r_i are constants that can be derived from constraints, as we will show. We can then express (24) in terms of the $\overline{x}_i$ as

$$V(\boldsymbol{x}) = \sum_{ij} a_{ij}[\overline{x}_i\overline{x}_j + (\overline{x}_i r_j + \overline{x}_j r_i) + r_i r_j] + \sum_i b_i(\overline{x}_i + r_i). \tag{30}$$

The linear terms in (30) are now

$$\sum_i \left\{\left[\sum_j (a_{ij}r_j + a_{ji}r_j)\right] + b_i\right\}\overline{x}_i \tag{31}$$

and they will vanish if the r_j's satisfy

$$b_i + \sum_j (a_{ij} + a_{ji})r_j = 0 \quad \text{or}$$
$$b_i + 2\sum_j a_{ij}r_j = 0, \quad i = 1,\dots,N. \tag{32}$$

The N Eqs. (32) can be solved for the r_j's because the condition in Eq. (28) holds. The constant term $(\sum_i b_i r_i) + (\sum_{ij} a_{ij} r_i r_j)$ in the potential V can be ignored and eliminated: Since the potential enters the NLFP equation only through its gradient, this constant term has no physical significance. Therefore, in terms of the shifted coordinates $\overline{x}_i$, we have

$$V(\overline{\boldsymbol{x}}) = \sum_{ij} a_{ij}\overline{x}_i\overline{x}_j, \tag{33a}$$

$$\tilde{K}_i(\overline{\boldsymbol{x}}) = \sum_j (c_{ij}\overline{x}_j) + \overline{d}_i, \tag{33b}$$

where $\overline{d}_i = \sum_j (c_{ij}r_j) + d_i$. We thus see that, after an appropriate shift in the phase space variables, the problem reduces to that of a homogeneous quadratic potential.

If the associated nonlinear Fokker-Planck equation admits a q-maxent stationary solution, even for one single value of β, it follows from (27a) that we must have

$$\sum_j c_{jj} = 0 \Rightarrow \nabla \cdot \tilde{\boldsymbol{K}} = 0, \tag{34}$$

from which it follows that the condition $\tilde{\boldsymbol{K}} \cdot \nabla V = 0$ also follows. In other words, for a quadratic potential V and a linear drift $\tilde{\boldsymbol{K}}$, if one has a q-maxent stationary solution even for one single value of β, it is possible after a coordinate shift to recast the system in terms of a drift field, complying with conditions (16) and (17).

VI. TWO-DIMENSIONAL SYSTEM WITH EXACT TIME-DEPENDENT *q*-GAUSSIAN SOLUTIONS

We now consider, as an example of a time-dependent solution of a nonlinear Fokker-Planck equation with a $\tilde{\boldsymbol{K}}$ not arising from a potential that admits a q-maxent stationary solution, a two-dimensional system submitted to the following quadratic potential and nongradient linear drift term. For simplicity of notation, we will name the phase space state variables $x \equiv \overline{x}_1$ and $y \equiv \overline{x}_2$, so the potential and drift term can be expressed as

$$V(\overline{\boldsymbol{x}}) = a(x^2 + y^2), \tag{35}$$

$$\tilde{\boldsymbol{K}}(\overline{\boldsymbol{x}}) = (-by, +bx). \tag{36}$$

It is well known that most nonlinear differential equations in physics, biology, and related areas do not admit general analytical solutions. In some cases, however, one is fortunate enough to have at least a particular analytical exact solution. As we will presently show, this is what happens with the nonlinear Fokker-Planck equation associated with the above potential and drift field. No general analytical solution is available, but it is possible to obtain a particular, exact, time-dependent solution of the q-Gaussian form. Particular analytical solutions of nonlinear differential equations are, for a variety of reasons, of considerable value. They provide concrete examples, where the kind of behavior exhibited by the solutions can be studied in a detailed and transparent form (even though the general solutions may have a much richer dynamics). Particular analytical solutions may constitute useful starting points for the construction of more general, analytical, approximate solutions. They are also useful to test the accuracy of numerical approaches for solving the nonlinear equations under consideration. The case of dynamical equations admitting exact q-Gaussian solutions is especially relevant because q-Gaussian densities are actually observed in nature, in diverse physical and biological scenarios [18], making it imperative to identify and investigate all the different dynamical mechanisms that may lead to q-Gaussian solutions.

It can be verified that (35) and (36) satisfy conditions given by Eqs. (16) and (17). The NLFP equation then has the form

$$\frac{\partial F}{\partial t} = D\nabla^2[F^{2-q}] + \frac{\partial[(2ax + by)F]}{\partial x} + \frac{\partial[(2ay - bx)F]}{\partial y}. \tag{37}$$

We propose the ansatz

$$F(x,y,t) = \eta(t)\{1 - (1-q)[\alpha(t)x^2 + \delta(t)xy + \gamma(t)y^2]\}_+^{1/(1-q)}, \tag{38}$$

where $\eta(t)$, $\alpha(t)$, $\delta(t)$, and $\gamma(t)$ are time-dependent parameters. This ansatz has a time-dependent, Tsallis, q-maximum entropy (q-maxent) form, with the time dependence represented in the parameters η, α, δ, and γ. We then define

$$\varphi = 1 - (1-q)(\alpha x^2 + \delta xy + \gamma y^2), \tag{39}$$

calculate the terms of the nonlinear Fokker-Planck equation (1), and obtain the expressions

$$\frac{\partial F}{\partial t} = \dot{\eta}\varphi^{1/(1-q)} - \eta(\dot{\alpha}x^2 + \dot{\delta}xy + \dot{\gamma}y^2)\varphi^{q/(1-q)}, \tag{40a}$$

$$\frac{\partial^2 F^{2-q}}{\partial x^2} = (2-q)\eta^{2-q}[-2\alpha\varphi^{1/(1-q)} + (2\alpha x + \delta y)^2\varphi^{q/(1-q)}], \tag{40b}$$

$$\frac{\partial^2 F^{2-q}}{\partial y^2} = (2-q)\eta^{2-q}[-2\gamma\varphi^{1/(1-q)} + (2\gamma y + \delta x)^2\varphi^{q/(1-q)}], \tag{40c}$$

$$\frac{\partial[(2ax+by)F]}{\partial x} = \eta[2a\varphi^{1/(1-q)} - (2ax+by) \times(2\alpha x + \delta y)\varphi^{q/(1-q)}], \tag{40d}$$

$$\frac{\partial[(2ay-bx)F]}{\partial y} = \eta[2a\varphi^{1/(1-q)} - (2ay-bx) \times(2\gamma y + \delta x)\varphi^{q/(1-q)}]. \tag{40e}$$

Next we substitute the right-hand side of Eqs. (40) into the NLFP equation (37) and, with some algebra, obtain the following set of ordinary differential equations for the time evolution of the parameters η, α, δ, and γ:

$$\frac{d\eta}{dt} = 4\eta a - 2(2-q)D\eta^{2-q}(\alpha+\gamma), \tag{41a}$$

$$\frac{d\alpha}{dt} = -(2-q)D\eta^{1-q}(4\alpha^2+\delta^2) + 4a\alpha - b\delta, \tag{41b}$$

$$\frac{d\gamma}{dt} = -(2-q)D\eta^{1-q}(4\gamma^2+\delta^2) + 4a\gamma + b\delta, \tag{41c}$$

$$\frac{d\delta}{dt} = -4(2-q)D\eta^{1-q}\delta(\alpha+\gamma) + 4a\delta + 2b(\alpha-\gamma). \tag{41d}$$

Therefore, the q-maxent ansatz (38) will be a solution of the NLFP equation (37), provided the functions $\eta(t)$, $\alpha(t)$, $\delta(t)$, and $\gamma(t)$ satisfy the set of four coupled ordinary differential equations (41).

When we interpret a solution $F(x_1,\ldots,x_N,t)$ of the NLFP equation (1) as a probability density in phase space or as a physical density of particles or other entities we should require that the norm I of F is finite, so that

$$I = \int F\, dx_1 dx_2 \cdots dx_N \leqslant \infty. \tag{42}$$

For the particular density function (38) to have a finite norm, the quadratic form $\alpha x^2 + \delta xy + \gamma y^2$ has to be positive definite. This guarantees that the curves of constant density (isodensity curves), given by $\alpha x^2 + \delta xy + \gamma y^2 = \text{const} > 0$, correspond to ellipses. For $q < 1$, the density (38) has a compact support: It has values different from zero, within a region limited by the cutoff boundary given by the ellipse, $\alpha x^2 + \delta xy + \gamma y^2 = \frac{1}{1-q}$. At this boundary, and outside it, the density vanishes. To have elliptic isodensity curves, the discriminant

$$\varsigma = \alpha\gamma - \frac{\delta^2}{4} \tag{43}$$

has to be positive. It follows from (41) that the time derivative of the discriminant is

$$\frac{d\varsigma}{dt} = [\delta^2 - 4\alpha\gamma][(2-q)D\eta^{1-q}(\alpha+\gamma) - 2a] = 4\varsigma[2a - (2-q)D\eta^{1-q}(\alpha+\gamma)]. \tag{44}$$

We see that the value of the discriminant is not constant in time. However, Eq. (44) implies that the positive character of ς is preserved during the time evolution of the system. For the proposed q-statistical ansatz (38), we find, after some algebra, that, given a positive value of the discriminant (43), the norm (42) is finite for $q < 2$ and is equal to

$$I = \frac{\pi\eta}{(2-q)\sqrt{\alpha\gamma - \frac{\delta^2}{4}}}. \tag{45}$$

Therefore, the allowed values of the q parameter, for the system considered in this section, are those smaller than 2.

After some more calculation, it is also possible to verify, using the equations of motion (41), that

$$\frac{dI}{dt} = \frac{\partial I}{\partial\eta}\frac{d\eta}{dt} + \frac{\partial I}{\partial\alpha}\frac{d\alpha}{dt} + \frac{\partial I}{\partial\gamma}\frac{d\gamma}{dt} + \frac{\partial I}{\partial\delta}\frac{d\delta}{dt} = 0, \tag{46}$$

so that I is a conserved quantity during the time evolution of the system, as is to be expected. Let us also note that, at a given time, all the isodensity curves of the density (38) are ellipses having the same eccentricity,

$$e = \sqrt{\frac{2\sqrt{(\alpha-\gamma)^2+\delta^2}}{\alpha+\gamma+\sqrt{(\alpha-\gamma)^2+\delta^2}}}. \tag{47}$$

The stationary solutions of the set of equations (41) determine the values of the parameters $\eta_{\rm stat}$, $\alpha_{\rm stat}$, $\delta_{\rm stat}$, and $\gamma_{\rm stat}$, in the ansatz (38), corresponding to stationary solutions of the NLFP equation (37). These stationary solutions are

$$\eta_{\rm stat} = \left[\frac{a}{(2-q)D\alpha_{\rm stat}}\right]^{1/(1-q)}, \quad \gamma_{\rm stat} = \alpha_{\rm stat}, \quad \delta_{\rm stat} = 0. \tag{48}$$

We see that the stationary solution is not unique. In fact, Eqs. (48) determine a monoparametric family of stationary solutions, parametrized by $\alpha_{\rm stat}$. That is, for different values of this parameter one obtains, through (48), different stationary solutions. Each of these solutions has a different normalization, given by [see Eq. (45)]

$$I = \frac{\pi\eta_{\rm stat}}{(2-q)\alpha_{\rm stat}}. \tag{49}$$

As we have already discussed, the norm I is preserved by the time evolution. Combining (48) with (49), one can express the stationary solution in terms of the norm I by

$$\begin{aligned}\eta_{\text{stat}} &= \left[\frac{aI}{\pi D}\right]^{1/(2-q)},\\ \alpha_{\text{stat}} &= \frac{\pi}{(2-q)I}\left[\frac{aI}{\pi D}\right]^{1/(2-q)},\\ \gamma_{\text{stat}} &= \alpha_{\text{stat}},\\ \delta_{\text{stat}} &= 0.\end{aligned} \tag{50}$$

It transpires from the above equations that the parameters η_{stat} and α_{stat} increase monotonically with a and decrease with D. This corresponds to the fact that the stationary density becomes more localized when the strength of the confining potential (determined by the parameter a) increases, while it tends to delocalize when the diffusion effects (characterized by the constant D) become larger. The stationary density F_q, expressed as a function of the radial variable $r = (x^2+y^2)^{1/2}$, is

$$F_q(r) = \eta_{\text{stat}}[1-(1-q)\alpha_{\text{stat}} r^2]_+^{1/(1-q)}. \tag{51}$$

These stationary distributions are of the q-exponential form (4). If one considers only solutions normalized to 1 then, for each value of q, one has only one stationary distribution, with a q-dependent β given by

$$\beta_q = \frac{\pi}{(2-q)Ia}\left[\frac{aI}{\pi D}\right]^{1/(2-q)}. \tag{52}$$

Models having a q-dependent β arise in the description of some phenomena, such as the interoccurrence times between losses in financial markets [see Fig. 3(b) in [29]], and also in connection with the role of dimensionality in complex networks (see Fig. 7 in [30]).

A density $F(\boldsymbol{x},t)$, governed by the partial differential equation (37), can be interpreted as describing the distribution of a set of particles interacting via short-range interactions, performing overdamped motion under the drag effects due to a uniformly rotating medium, and confined by an external harmonic potential. To see this, let us consider the equation of motion of one individual test particle of this system

$$m\ddot{\boldsymbol{r}} = -\nabla W_{\text{int}} - \nabla W_{\text{ext}} - \Gamma(\dot{\boldsymbol{r}} - \dot{\boldsymbol{r}}_R), \tag{53}$$

where m is the mass of the test particle, W_{int} is the potential function associated with the forces acting on the test particle due to the other particles of the system, W_{ext} is the external confining potential, and Γ is a drag coefficient describing the drag forces due to a resisting medium, which rotates uniformly with an angular velocity Ω. Notice that the equation of motion (53) is expressed with respect to an inertial reference frame [with Cartesian coordinates (x,y)] and not with respect to the rotating frame where the resisting medium is at rest. With respect to the inertial frame, the local velocity $\dot{\boldsymbol{r}}_R$ of the medium has components $(-\Omega y, +\Omega x)$.

Since the interactions between the particles are short range, we assume that the potential function W_{int} is a function of the local density F, that is, $W_{\text{int}} = \mathcal{D}(F)$. In the regime of overdamped motion, Eq. (53) becomes

$$\dot{\boldsymbol{r}} = -\frac{1}{\Gamma}\nabla W_{\text{int}} - \frac{1}{\Gamma}\nabla W_{\text{ext}} + \dot{\boldsymbol{r}}_R, \tag{54}$$

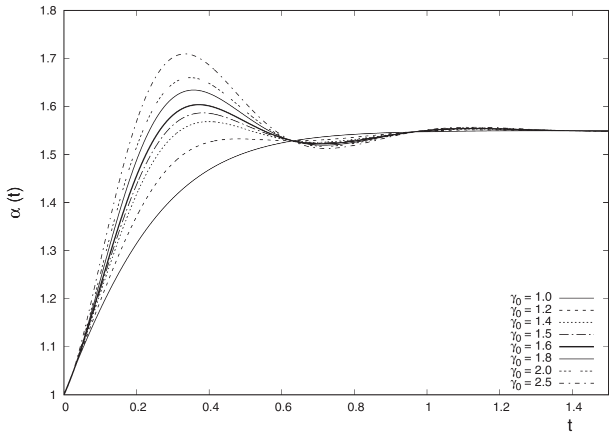

FIG. 1. Evolution of the parameter α appearing in the time-dependent solution (38) of the NLFP equation (37) for $q = 0.5$. The units employed are defined in terms of the constants D and b appearing in the NLFP equation. The parameter α has dimensions of inverse squared length and is measured in units of $\frac{b}{8D}$. The time t is measured in units of $\frac{4}{b}$.

implying that the velocity $\dot{\boldsymbol{r}}$ of a particle in the system, at a given time, is completely determined by its location $\boldsymbol{r}$. It can then be verified, after some calculations, that the continuity equation in configuration space $\partial F/\partial t = -\nabla(\dot{\boldsymbol{r}}F)$, describing the evolution of the space density F of a set of particles moving according to the equation of motion (54), is precisely the NLFP equation (37), after the identifications $\frac{\mathcal{D}}{\Gamma} = \frac{2-q}{1-q}DF^{1-q}$, $\frac{W_{\text{ext}}}{\Gamma} = a\boldsymbol{r}^2$, and $b = \Omega$.

An illustrative example of the time evolution of the q-Gaussian solution (38) is provided in Figs. 1–4. In these figures, the parameters α, γ, δ, and η, determining the evolving size and shape of the two-dimensional q-Gaussian solution (38), are depicted as a function of time. The different curves shown in each figure correspond to the NLFP equation (37), with $q = 0.5$, $D = 0.5$, $a = 1$, $b = 4$, and different initial

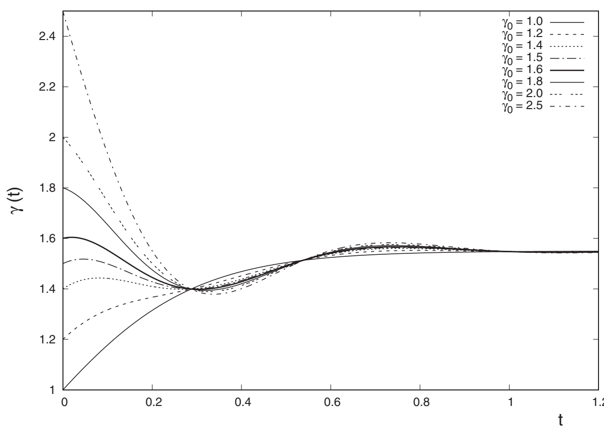

FIG. 2. Evolution of the parameter γ appearing in the time-dependent solution (38) of the NLFP equation for $q = 0.5$. The parameter γ has dimensions of inverse squared length. The units employed are the same as in Fig. 1.

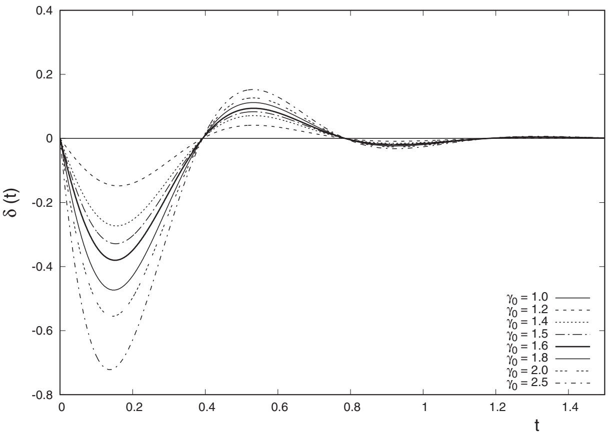

FIG. 3. Evolution of the parameter δ appearing in the time-dependent solution (38) of the NLFP equation for $q = 0.5$. The parameter δ has dimensions of inverse squared length. The units employed are the same as in Fig. 1.

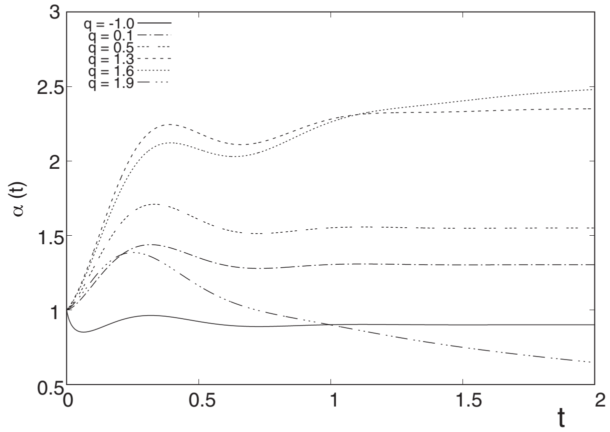

FIG. 5. Evolution, for different q values, of the parameter α appearing in the time-dependent solution (38) of the NLFP equation. Dimensions and units are the same as in Fig. 1.

conditions. The curves were therefore obtained from the numerical integration of the set of coupled ordinary differential equations (41). All solutions exhibited correspond to evolving densities normalized to unity [that is, $I = 1$ in Eq. (45)]. The initial conditions are $\alpha_0 = 1$ and $\delta_0 = 0$, with different initial values of the parameter γ, as indicated in the figures. The initial value of η is calculated from the initial values of the other three parameters, using the normalization condition $I = 1$.

It can be appreciated from Figs. 1–4 that the different initial densities considered (all having the same norm $I = 1$) relax to the same final stationary distribution (characterized by the same value of the norm). This stationary distribution is rotationally symmetric. Consequently, the initial asymmetry of the density tends to decrease as the evolution takes place (the two axis of the isodensity curves tend to become equal to each other). The oscillatory behavior of the parameter δ, which takes alternating signs as time advances, indicates that the asymmetric density rotates as the evolution proceeds. Note that, at the times when $\delta = 0$, the axis of the isodensity curves is parallel to the coordinate axis. This happens at approximately regular time intervals, indicating that the elliptical isodensity curves rotate at an approximately constant mean angular velocity. The oscillatory behavior associated with the rotation affects the other variables (besides δ) as well, which also exhibit oscillations whose amplitudes tend to decrease as the density function F relaxes towards the stationary one.

We solved numerically the set of differential equations (41) for different values of q, both smaller and larger than 1, and found that the qualitative features of the corresponding time evolution share, for all q values, some basic similarities. The parameters α, δ, and γ always show first an oscillatory transient and then they relax towards their stationary values. To illustrate this, we show in Fig. 5 the evolution of the parameter α for different values of q and the initial condition $\alpha_0 = 1$, $\delta_0 = 0$, and $\gamma_0 = 2.5$. In all cases, we consider an initial distribution

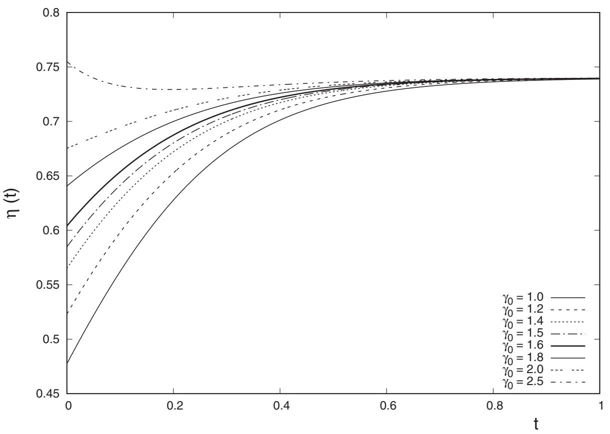

FIG. 4. Evolution of the parameter η appearing in the time-dependent solution (38) of the NLFP equation for $q = 0.5$. The parameter η is dimensionless. The time t is measured in the same units as in Fig. 1.

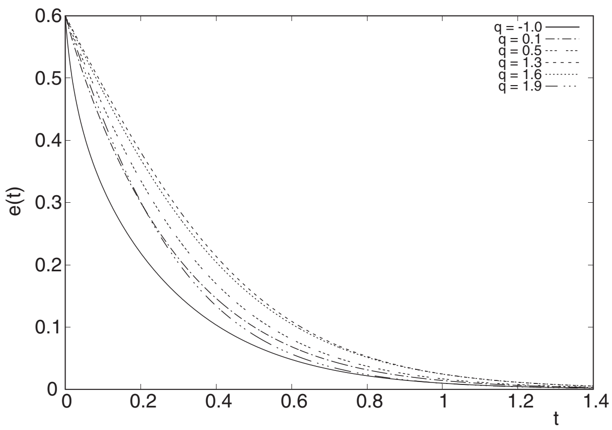

FIG. 6. Evolution, for different q values, of the eccentricity of the isodensity curves, of the time-dependent solution (38) of the NLFP equation. The eccentricity is dimensionless. The time t is measured in the same units as in Fig. 1.

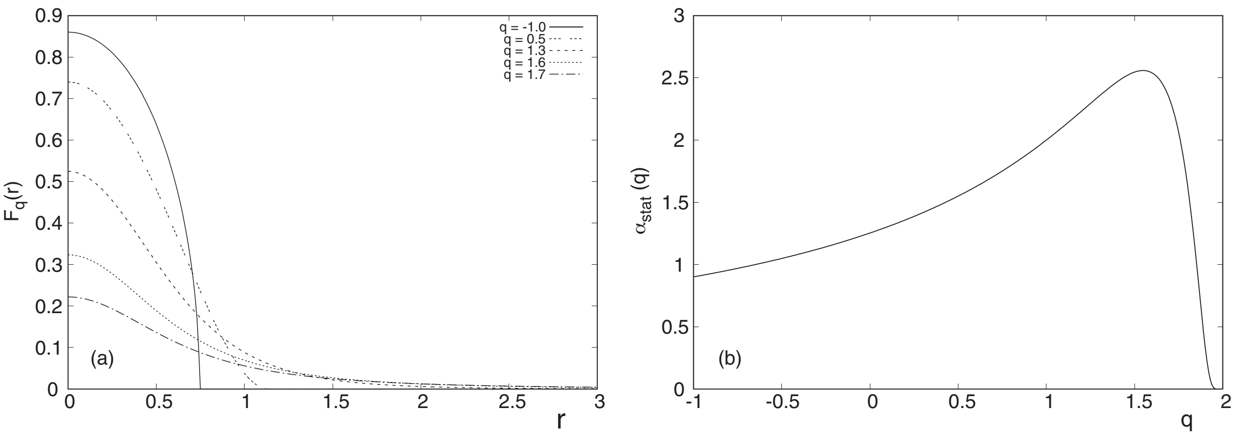

FIG. 7. (a) Stationary distribution F_q as a function of r for different values of the parameter q. The quantity F_q is dimensionless and the coordinate r is measured in units $\sqrt{\frac{8D}{b}}$. (b) Stationary value of the parameter α as a function of q. The units for α are as in Fig. 1 and q is dimensionless.

normalized to one, this condition determining the initial value of the parameter η.

The evolution of the eccentricity of the isodensity curves is shown, for various q values, in Fig. 6. It is clear that, in all cases, the eccentricity decreases monotonically with time, consistently with the fact that the time-dependent density (38) relaxes towards the stationary density $F_q(r)$ [see Eq. (51)], which is rotationally symmetric. Our numerical results indicate that the eccentricity of the isodensity curves decays in an asymptotically exponential way. The stationary distribution $F_q(r)$ is depicted in Fig. 7(a) for different q values. Figure 7(b) depicts the stationary value α_{stat} of the parameter α appearing in the solution (38) as a function of q. We see that α_{stat} is not a monotonic function of q; it has a maximum value around $q = 1.55$. This helps to understand some aspects of Fig. 5.

Several features of the curves $\alpha(t)$ depicted in Fig. 5 are determined by the relationship between the initial value α_0, which is the same for all these curves, and the asymptotic q-dependent stationary values $\alpha_{stat}(q)$ to which α tends at large times. For instance, for $q = -1$, the stationary value $\alpha_{stat}(-1)$ is close to the initial condition α_0, common to all the cases considered in Fig. 5. Consequently, for $q = -1$, α describes oscillations of small amplitude and quickly relaxes to its stationary value. On the other hand, for increasing q values up to $q \approx 1.55$, $\alpha_{stat}(q)$ increases with q [see Fig. 7(b)] and the corresponding curves in Fig. 5 exhibit initial oscillations of higher amplitude and later relax to values of $\alpha_{stat}(q)$ that are larger than the initial value α_0. Finally, for q values larger than $q \approx 1.55$, the stationary $\alpha_{stat}(q)$ decreases quickly with q. For instance, for $q = 1.9$, the stationary value of α is well below the initial value α_0. This explains why the $\alpha(t)$ curve corresponding to $q = 1.9$ crosses the already mentioned one corresponding to $\alpha = -1$.

VII. CONCLUSION

We investigated the main properties of multidimensional NLFP equations, involving curl drift forces. We considered drift force fields comprising both an irrotational term $\boldsymbol{G}$, derived from a potential function $V(\boldsymbol{x})$, and a curl part, the nongradient term $\tilde{\boldsymbol{K}}$. We determined the requirements that the two parts $\boldsymbol{G}$ and $\tilde{\boldsymbol{K}}$ of the drift field have to satisfy in order for the corresponding NLFP equation to admit a stationary solution of the q-maxent form [that is, a q exponential of the potential function $V(\boldsymbol{x})$, associated with the gradient component of the drift force]. We found that this kind of stationary solution exists for a continuous range of values of the parameter β if and only if the curl part $\tilde{\boldsymbol{K}}$ is divergence-free and the curl part is orthogonal to the gradient part $\boldsymbol{G}$. We also proved that NLFP equations admitting a stationary solution verify an H theorem in terms of an appropriate linear combination of the S_q entropic functional and the mean value of the potential V. Finally, we studied exact, analytical, time-dependent solutions of a two-dimensional NLFP equation, describing a system of interacting particles in an overdamped motion regime under the drag effects originating on a uniformly rotating medium. The connection between rotation and NLFP equations with curl forces, combined with the connection between q thermostatistics and self-gravitating systems, indicates that those evolution equations may have applications in geophysical and astrophysical problems. Previous successful physical applications of NLFP equations also suggest that experimental implementations involving rotating granular materials may also be worth exploring.

Another potential field of application of the NLFP dynamics investigated in the present work is the space-time behavior of some biological systems [31]. Diffusion processes are useful to model the spread of biological populations [32,33]. Nonlinear diffusion equations have been proposed as effective descriptions of the interaction between the members of a diffusing biological population [34–36]. On the other hand, drift terms can be used to describe other effects on the motion of the individuals. In this biological context, since the forces are not fundamental but rather the effective result of a set of complex circumstances, it is to be expected that nongradient forces can be relevant. Nonlinear Fokker-Planck equations with nongradient drift fields may also be useful in connection with the generalized Boltzmann machine approach (based on a q generalization of simulated annealing [37]) to neural network models of memory [38] when considering asymmetric neural interactions. Any further developments along these or related lines will be very welcome.

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