

# On a $q$ -Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

**Abstract.** The standard central limit theorem plays a fundamental role in Boltzmann-Gibbs statistical mechanics. This important physical theory has been generalized [1] in 1988 by using the entropy  $S_q = \frac{1 - \sum_i p_i^q}{q-1}$  (with  $q \in \mathcal{R}$ ) instead of its particular BG case  $S_1 = S_{BG} = -\sum_i p_i \ln p_i$ . The theory which emerges is usually referred to as *nonextensive statistical mechanics* and recovers the standard theory for  $q = 1$ . During the last two decades, this  $q$ -generalized statistical mechanics has been successfully applied to a considerable amount of physically interesting complex phenomena. A conjecture[2] and numerical indications available in the literature have been, for a few years, suggesting the possibility of  $q$ -versions of the standard central limit theorem by allowing the random variables that are being summed to be strongly correlated in some special manner, the case  $q = 1$  corresponding to standard probabilistic independence. This is what we prove in the present paper for  $1 \leq q < 3$ . The attractor, in the usual sense of a central limit theorem, is given by a distribution of the form  $p(x) = C_q [1 - (1 - q)\beta x^2]^{1/(1-q)}$  with  $\beta > 0$ , and normalizing constant  $C_q$ . These distributions, sometimes referred to as  $q$ -Gaussians, are known to make, under appropriate constraints, extremal the functional  $S_q$  (in its continuous version). Their  $q = 1$  and  $q = 2$  particular cases recover respectively Gaussian and Cauchy distributions.

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## 1. Introduction

Limit theorems and, particularly, the central limit theorems (CLT), surely are among the most important theorems in probability theory and statistics. They play an essential role in various applied sciences, including statistical mechanics. Historically A. de Moivre, P.S. de Laplace, S.D. Poisson and C.F. Gauss have first shown that Gaussian is the attractor of independent systems with a finite second variance. Chebyshev, Markov, Liapounov, Feller, Lindeberg, Lévy have contributed essentially to the development of the central limit theorem.

It is well known in the classical Boltzmann-Gibbs (BG) statistical mechanics that the Gaussian maximizes, under appropriate constraints, the Boltzmann-Gibbs entropy  $S_{BG} = -\sum_i p_i \ln p_i$  ( $S_{BG} = -\int dx p(x) \ln p(x)$  in its continuous form). The  $q$ -generalization of the classic entropy introduced in [1] as the basis for generalizing the BG theory, and given by  $S_q = \frac{1-\sum_i p_i^q}{q-1}$  ( $S_q = \left[1 - \int dx [p(x)]^q\right]/(q-1)$  in its continuous form), with  $q \in \mathcal{R}$  and  $S_1 = S_{BG}$ , reaches its maximum at the distributions usually referred to as  $q$ -Gaussians (see [3]). This fact, and a number of conjectures [2], numerical indications [4], and the content of some other studies [1, 3, 5, 6] suggest the existence of a  $q$ -analog of the CLT as well. In this paper we prove a generalization of the CLT for  $1 \leq q < 3$ . The case  $q < 1$  requires essentially different technique, therefore we leave it for a separate paper.

In the classical CLT, the random variables are required to be independent. Central limit theorems were established for weakly dependent random variables also. An introduction to this area can be found in [7, 8, 9, 10, 11, 12] (see also references therein), where different types of dependence are considered, as well as the history of the developments. The CLT does not hold if correlation between far-ranging random variables is not neglectable (see [13]). Nonextensive statistical mechanics deals with strongly correlated random variables, whose correlation does *not* rapidly decrease with increasing 'distance' between random variables localized or moving in some geometrical lattice (or continuous space) on which a 'distance' can be defined. This type of correlation is sometimes referred to as *global* correlation (see [17] for more details).

In general,  $q$ -CLT appears to be untractable if we rely on the classic algebra. However, nonextensive statistical mechanics uses constructions based on a special algebra sometimes referred to as  $q$ -algebra, which depends on parameter  $q$ . We show that, in the framework of  $q$ -algebra, the

corresponding  $q$ -generalization of the central limit theorem becomes possible and relatively simple.

The theorems obtained in this paper are represented as a series of theorems depending on the type of correlations. We will consider three types of correlation. An important distinction from the classic CLT is the fact that a complete formulation of  $q$ -CLT is not possible within only one given  $q$ . The parameter  $q$  is connected with two other numbers,  $q_* = z^{-1}(q)$  and  $q^* = z(q)$ , where  $z(s) = (1+s)/(3-s)$ . We will see that  $q_*$  identifies an attractor, while  $q^*$  yields the scaling rate. In general, the present  $q$ -generalization of the CLT is connected with a triplet  $(q_{k-1}, q_k, q_{k+1})$  determined by a given  $q \in [1, 2)$ . For systems having correlation identified by  $q_k$ , the index  $q_{k-1}$  determines the attracting  $q$ -Gaussian, while the index  $q_{k+1}$  indicates the scaling rate. Note that, if  $q = 1$ , then the entire family of theorems reduces to one element, thus recovering the classic CLT.

The paper is organized as follows. In Section 2 we recall the basics of  $q$ -algebra, definitions of  $q$ -exponential and  $q$ -logarithm. Then we introduce a transform  $F_q$  and study its basic properties. For  $q \neq 1$   $F_q$  is not a linear operator. Note that  $F_q$  is linear only if  $q = 1$  and in this case coincides with the classic Fourier transform. Lemma 2.5 implies that  $F_q$  is invertible in the class of  $q$ -Gaussians. An important property of  $F_q$  is that it maps a  $q$ -Gaussian to a  $q^*$ -Gaussian (with a constant factor). In Section 3 we prove the main result of this paper, i.e., the  $q$ -version of the CLT. We introduce the notion of  $q$ -independent random variables (three types), which classify correlated random variables. Only in the case  $q = 1$  the correlation disappears, thus recovering the classic notion of independence of random variables.

## 2. $q$ -Algebra and $q$ -Fourier Transform

### 2.1. $q$ -sum and $q$ -product

The basic operations of the  $q$ -algebra appear naturally in nonextensive statistical mechanics. It is well known that, if  $A$  and  $B$  are two independent subsystems, then the total BG entropy satisfies the additivity property

$$S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B).$$

Additivity is not preserved for  $q \neq 1$ . Indeed, we easily verify [1, 3, 2]

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q) S_q(A) S_q(B).$$

Introduce the  $q$ -sum of two numbers  $x$  and  $y$  by the formula

$$x \oplus_q y = x + y + (1 - q)xy.$$

Then, obviously,  $S_q(A + B) = S_q(A) \oplus_q S_q(B)$ . It is readily seen that the  $q$ -sum is commutative, associative, recovers the usual summing operation if  $q = 1$  (i.e.  $x \oplus_1 y = x + y$ ), and preserves 0 as the neutral element (i.e.  $x \oplus_q 0 = x$ ). By inversion, we can define the  $q$ -subtraction as  $x \ominus_q y = \frac{x-y}{1+(1-q)y}$ . Further, the  $q$ -product for  $x, y$  is defined by the binary relation  $x \otimes_q y = [x^{1-q} + y^{1-q} - 1]_+^{\frac{1}{1-q}}$ . The symbol  $[x]_+$  means that  $[x]_+ = x$ , if  $x \geq 0$ , and  $[x]_+ = 0$ , if  $x \leq 0$ . Also this operation is commutative, associative, recovers the usual product when  $q = 1$  (i.e.  $x \otimes_1 y = xy$ ), preserves 1 as the unity (i.e.  $x \otimes_q 1 = x$ ). Again, by inversion, a  $q$ -division can be defined:  $x \oslash_q y = [x^{1-q} - y^{1-q} + 1]_+^{\frac{1}{1-q}}$ . Note that, for  $q \neq 1$ , division by zero is allowed. As we will see below, the  $q$ -sum and the  $q$ -product are connected each other through the  $q$ -exponential, generalizing the fundamental property of the exponential function,  $e^{x+y} = e^x e^y$  (or  $\ln(xy) = \ln x + \ln y$  in terms of logarithm).

## 2.2. $q$ -exponential and $q$ -logarithm

The  $q$ -analysis relies essentially on the analogs of exponential and logarithmic functions, which are called  $q$ -exponential and  $q$ -logarithm [18]. In the mathematical literature there are other generalizations of the classic exponential distinct from the  $q$ -exponential used in the present paper. These generalizations were introduced by Euler [14], Jackson [15], and others. See [16] for details.

The  $q$ -exponential and  $q$ -logarithm, which are denoted by  $e_q^x$  and  $\ln_q x$ , are respectively defined as  $e_q^x = [1 + (1-q)x]_+^{\frac{1}{1-q}}$  and  $\ln_q x = \frac{x^{1-q} - 1}{1-q}$ , ( $x > 0$ ).

For the  $q$ -exponential, the relations  $e_q^{x \oplus_q y} = e_q^x e_q^y$  and  $e_q^{x+y} = e_q^x \otimes_q e_q^y$  hold true. These relations can be rewritten equivalently as follows:  $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$ , and  $\ln_q(xy) = \ln_q x \oplus_q \ln_q y^1$ . The  $q$ -exponential and  $q$ -logarithm have asymptotics  $e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2)$ ,  $x \rightarrow 0$  and  $\ln_q(1+x) = x - \frac{q}{2}x^2 + o(x^2)$ ,  $x \rightarrow 0$ . For  $q \neq 1$ , we can define  $e_q^{iy}$ , where  $i$  is the imaginary unit and  $y$  is real, as the principal value of  $[1 + i(1-q)y]_+^{\frac{1}{1-q}}$ ,

<sup>1</sup>These properties reflect the possible extensivity of the nonadditive entropy  $S_q$  in the presence of special correlations [17, 24, 26, 28, 29].

namely

$$e_q^{iy} = [1 + (1 - q)^2 y^2]^{\frac{1}{2(1-q)}} e^{\frac{i \arctan[(q-1)y]}{q-1}}, \quad q \neq 1.$$

Analogously, for  $z = x + iy$  we can define  $e_q^z = e_q^{x+iy}$ , which is equal to  $e_q^x \otimes_q e_q^{iy}$  in accordance with the property mentioned above. Note that, if  $q < 1$ , then for real  $y$ ,  $|e_q^{iy}| \geq 1$  and  $|e_q^{iy}| \sim (1 + y^2)^{\frac{1}{2(1-q)}}$ ,  $x \rightarrow \infty$ . Similarly, if  $q > 1$ , then  $0 < |e_q^{iy}| \leq 1$  and  $|e_q^{iy}| \rightarrow 0$  if  $|y| \rightarrow \infty$ .

### 2.3. $q$ -Gaussian

Let  $\beta$  be a positive number. We call the function

$$G_q(\beta; x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}. \tag{2.1}$$

a  $q$ -Gaussian;  $C_q$  is the normalizing constant, namely  $C_q = \int_{-\infty}^{\infty} e_q^{-x^2} dx$ . It is easy to verify that

$$C_q = \begin{cases} \frac{2}{\sqrt{1-q}} \int_0^{\pi/2} (\cos t)^{\frac{3-q}{1-q}} dt = \frac{2\sqrt{\pi} \Gamma\left(\frac{1}{1-q}\right)}{(3-q)\sqrt{1-q} \Gamma\left(\frac{3-q}{2(1-q)}\right)}, & -\infty < q < 1, \\ \sqrt{\pi}, & q = 1, \\ \frac{2}{\sqrt{q-1}} \int_0^{\infty} (1 + y^2)^{\frac{-1}{q-1}} dy = \frac{\sqrt{\pi} \Gamma\left(\frac{3-q}{2(q-1)}\right)}{\sqrt{q-1} \Gamma\left(\frac{1}{q-1}\right)}, & 1 < q < 3. \end{cases} \tag{2.2}$$

For  $q < 1$ , the support of  $G_q(\beta; x)$  is compact since this density vanishes for  $|x| > 1/\sqrt{(1-q)\beta}$ . Notice also that, for  $q < 5/3$  ( $5/3 \leq q < 3$ ), the variance is finite (diverges). Finally, we can easily check that there are relationships between different values of  $q$ . For example,  $e_q^{-x^2} = \left[ e_{2-\frac{1}{q}}^{-qx^2} \right]^{\frac{1}{q}}$ .

The following lemma establishes a general relationship (which contains the previous one as a particular case) between different  $q$ -Gaussians.

**Lemma 2.1.** *For any real  $q_1, \beta_1 > 0$  and  $\delta > 0$  there exist uniquely determined  $q_2 = q_2(q_1, \delta)$  and  $\beta_2 = \beta_2(\delta, \beta_1)$ , such that*

$$(e_{q_1}^{-\beta_1 x^2})^\delta = e_{q_2}^{-\beta_2 x^2}.$$

Moreover,  $q_2 = \delta^{-1}(\delta - 1 + q_1)$ ,  $\beta_2 = \delta\beta_1$ .

*Proof.* Let  $q_1 \in R^1, \beta_1 > 0$  and  $\delta > 0$  be any fixed real numbers. For the equation

$$(1 - (1 - q_1)\beta_1 x^2)^{\frac{\delta}{1-q_1}} = (1 - (1 - q_2)\beta_2 x^2)^{\frac{1}{1-q_2}}$$

to be an identity, it is needed  $(1 - q_1)\beta_1 = (1 - q_2)\beta_2$ ,  $1 - q_1 = \delta(1 - q_2)$ . These equations have a unique solution  $q_2 = \delta^{-1}(\delta - 1 + q_1)$ ,  $\beta_2 = \delta\beta_1$ .  $\square$

The set of all  $q$ -Gaussians with a positive constant factor will be denoted by  $\mathcal{G}_q$ , i.e.,

$$\mathcal{G}_q = \{bG_q(\beta, x) : b > 0, \beta > 0\}.$$

#### 2.4. $q$ -Fourier transform

From now on we assume that  $1 \leq q < 3$ . For these values of  $q$  we introduce the  $q$ -Fourier transform  $F_q$ , an operator which coincides with the Fourier transform if  $q = 1$ . Note that the  $q$ -Fourier transform is defined on the basis of the  $q$ -product and the  $q$ -exponential, and, in contrast to the usual Fourier transform, is a *nonlinear* transform for  $q \in (1, 3)$ . Let  $f$  be a non-negative measurable function with  $\text{supp } f \subseteq \mathbb{R}$ . The  $q$ -Fourier transform for  $f$  is defined by the formula

$$F_q[f](\xi) = \int_{\text{supp } f} e_q^{ix\xi} \otimes_q f(x) dx, \quad (2.3)$$

where the integral is understood in the Lebesgue sense. For discrete functions  $f_k, k \in \mathcal{Z} = \{0, \pm 1, \dots\}$ ,  $F_q$  is defined as

$$F_q[f](\xi) = \sum_{k \in \mathcal{Z}_f} e_q^{ik\xi} \otimes_q f_k, \quad (2.4)$$

where  $\mathcal{Z}_f = \{k \in \mathcal{Z} : f_k \neq 0\}$ .

In what follows we use the same notation in both cases. We also call (2.3) or (2.4) the  $q$ -characteristic function of a given random variable  $X$  with an associated density  $f(x)$ , using the notations  $F_q(X)$  or  $F_q(f)$  equivalently. The following lemma establishes the expression of the  $q$ -Fourier transform in terms of the standard product, instead of the  $q$ -product.

**Lemma 2.2.** *The  $q$ -Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) e_q^{ix\xi [f(x)]^{q-1}} dx. \quad (2.5)$$

*Proof.* For  $x \in \text{supp } f$  we have

$$\begin{aligned} e_q^{ix\xi} \otimes_q f(x) &= [1 + (1 - q)ix\xi + [f(x)]^{1-q} - 1]^{\frac{1}{1-q}} = \\ &= f(x)[1 + (1 - q)ix\xi [f(x)]^{q-1}]^{\frac{1}{1-q}}. \end{aligned} \quad (2.6)$$

Integrating both sides of Eq. (2.6) we obtain (2.5).  $\square$

Analogously, for a discrete  $f_k$ ,  $k \in \mathbb{Z}$ , (2.4) can be represented as

$$F_q[f](\xi) = \sum_{k \in \mathbb{Z}} f_k e_q^{ik\xi f_k^{q-1}}.$$

**Corollary 2.3.** *The  $q$ -Fourier transform exists for any nonnegative  $f \in L_1(\mathbb{R})$ . Moreover,  $|F_q[f](\xi)| \leq \|f\|_{L_1}$ .<sup>2</sup>*

*Proof.* This is a simple implication of Lemma 2.2 and of the asymptotics of  $e_q^{ix}$  for large  $|x|$  mentioned above.  $\square$

**Corollary 2.4.** *Assume  $f(x) \geq 0$ ,  $x \in \mathbb{R}$  and  $F_q[f](\xi) = 0$  for all  $\xi \in \mathbb{R}$ . Then  $f(x) = 0$  for almost all  $x \in \mathbb{R}$ .*

**Lemma 2.5.** *Let  $1 \leq q < 3$ . For the  $q$ -Fourier transform of a  $q$ -Gaussian, the following formula holds:*

$$F_q[G_q(\beta; x)](\xi) = \left( e_q^{-\frac{\xi^2}{4\beta^{2-q} C_q^{2(q-1)}}} \right)^{\frac{3-q}{2}}. \tag{2.7}$$

*Proof.* Denote  $a = \frac{\sqrt{\beta}}{C_q}$  and write

$$F_q[a e_q^{-\beta x^2}](\xi) = \int_{-\infty}^{\infty} (a e_q^{-\beta x^2}) \otimes_q (e_q^{ix\xi}) dx$$

using the property  $e_q^{x+y} = e_q^x \otimes_q e_q^y$  of the  $q$ -exponential, in the form

$$\begin{aligned} F_q[a e_q^{-\beta x^2}](\xi) &= a \int_{-\infty}^{\infty} e_q^{-\beta x^2 + ia^{q-1} x \xi} dx a \int_{-\infty}^{\infty} e_q^{-(\sqrt{\beta} x - \frac{ia^{q-1} \xi}{2\sqrt{\beta}})^2 - \frac{a^{2(q-1)} \xi^2}{4\beta}} dx \\ &= a \int_{-\infty}^{\infty} e_q^{-(\sqrt{\beta} x - \frac{ia^{q-1} \xi}{2\sqrt{\beta}})^2} \otimes_q e_q^{-\frac{a^{2(q-1)} \xi^2}{4\beta}} dx. \end{aligned}$$

The substitution  $y = \sqrt{\beta} x - \frac{ia^{q-1} \xi}{2\sqrt{\beta}}$  yields the equation

$$F_q[a e_q^{-\beta x^2}](\xi) = \frac{a}{\sqrt{\beta}} \int_{-\infty+i\eta}^{\infty+i\eta} e_q^{-y^2} \otimes_q e_q^{-\frac{a^{2(q-1)} \xi^2}{4\beta}} dy,$$

where  $\eta = \frac{\xi a^{q-1}}{2\sqrt{\beta}}$ . Moreover, the Cauchy theorem on integrals over closed curves is applicable because of at least a power-law decay of  $q$ -exponential

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<sup>2</sup>Here, and elsewhere,  $\|f\|_{L_1} = \int_{\mathcal{R}} f(x) dx$ ,  $L_1$  being the space of absolutely integrable functions.

for any  $1 \leq q < 3$ . By using it, we can transfer the integration from  $R + i\eta$  to  $R$ . Hence, applying again Lemma 2.2, we have

$$\begin{aligned} F_q[G_q(\beta; x)](\xi) &= \frac{ae_q^{-\frac{a^2(q-1)}{4\beta}\xi^2}}{\sqrt{\beta}} \int_{-\infty}^{\infty} e_q^{-y^2 \left( e_q^{-\frac{a^2(q-1)}{4\beta}\xi^2} \right)^{q-1}} dy \\ &= \frac{aC_q}{\sqrt{\beta}} \left( e_q^{-\frac{a^2(q-1)}{4\beta}\xi^2} \right)^{1-\frac{q-1}{2}}. \end{aligned}$$

Simplifying the last expression, we arrive at Eq. (2.7). □

Introduce the function  $z(s) = \frac{1+s}{3-s}$  for  $s \in (-\infty, 3)$ , and denote its inverse  $z^{-1}(t)$ ,  $t \in (-1, \infty)$ . It can be easily verified that  $z\left(\frac{1}{z(s)}\right) = \frac{1}{s}$  and  $z\left(\frac{1}{s}\right) = \frac{1}{z^{-1}(s)}$ . Let  $q_1 = z(q)$  and  $q_{-1} = z^{-1}(q)$ . It follows from the mentioned properties of  $z(q)$  that

$$z\left(\frac{1}{q_1}\right) = \frac{1}{q} \quad \text{and} \quad z\left(\frac{1}{q}\right) = \frac{1}{q_{-1}}. \tag{2.8}$$

The function  $z(s)$  also possess the following two important properties

$$z(s) z(2-s) = 1 \quad \text{and} \quad z(2-s) + z^{-1}(s) = 2. \tag{2.9}$$

It follows from these properties that  $q_{-1} + \frac{1}{q_1} = 2$ .

**Corollary 2.6.** *For  $q$ -Gaussians the following  $q$ -Fourier transforms hold*

$$F_q[G_q(\beta; x)](\xi) = e_{q_1}^{-\beta_*(q)\xi^2}, \quad q_1 = z(q), \quad 1 \leq q < 3; \tag{2.10}$$

$$F_{q_{-1}}[G_{q_{-1}}(\beta; x)](\xi) = e_q^{-\beta_*(q_{-1})\xi^2}, \quad q_{-1} = z^{-1}(q), \quad 1 \leq q < 3, \tag{2.11}$$

where  $\beta_*(s) = \frac{3-s}{8\beta^{2-s}C_s^{2(s-1)}}$  (or, more symmetrically,  $\beta_*^{\frac{1}{\sqrt{2-s}}}\beta^{\sqrt{2-s}} = K(s)$  with  $K(s) = \left[ \frac{3-s}{8C_s^{2(s-1)}} \right]^{\frac{1}{\sqrt{2-s}}}$ ;  $0 \leq K(s) < 1$  for  $s \leq 2$ , with  $\lim_{s \rightarrow -\infty} K(s) = K(2) = 0$ ).

*Remark 2.7.* Note that  $\beta_*(s) > 0$  if  $s < 3$ .

**Corollary 2.8.** *The following mappings*

$$F_q : \mathcal{G}_q \rightarrow \mathcal{G}_{q_1}, \quad q_1 = z(q), \quad 1 \leq q < 3,$$

$$F_{q_{-1}} : \mathcal{G}_{q_{-1}} \rightarrow \mathcal{G}_q, \quad q_{-1} = z^{-1}(q), \quad 1 \leq q < 3,$$

*hold and they are injective.*



**Corollary 2.9.** *There exist the following inverse  $q$ -Fourier transforms*

$$F_q^{-1} : \mathcal{G}_{q_1} \rightarrow \mathcal{G}_q, \quad q_1 = z(q), \quad 1 \leq q < 3,$$

$$F_{q^{-1}}^{-1} : \mathcal{G}_q \rightarrow \mathcal{G}_{q^{-1}}, \quad q^{-1} = z^{-1}(q), \quad 1 \leq q < 3.$$

**Lemma 2.10.** *The following mappings*

$$F_{\frac{1}{q_1}} : \mathcal{G}_{\frac{1}{q_1}} \rightarrow \mathcal{G}_{\frac{1}{q}}, \quad q_1 = z(q), \quad 1 \leq q < 3,$$

$$F_{\frac{1}{q}} : \mathcal{G}_{\frac{1}{q}} \rightarrow \mathcal{G}_{\frac{1}{q^{-1}}}, \quad q^{-1} = z^{-1}(q), \quad 1 \leq q < 3.$$

hold.

*Proof.* The assertion of this lemma follows from Corollary 2.8 taking into account the properties (2.8).  $\square$

Let us introduce the sequence  $q_n = z_n(q) = z(z_{n-1}(q)), n = 1, 2, \dots$ , with a given  $q = z_0(q), q < 3$ . We can extend the sequence  $q_n$  for negative integers  $n = -1, -2, \dots$  as well putting  $q_{-n} = z_{-n}(q) = z^{-1}(z_{1-n}(q)), n = 1, 2, \dots$ . It is not hard to verify that

$$q_n = \frac{2q + n(1 - q)}{2 + n(1 - q)} = 1 + \frac{2(q - 1)}{2 - n(q - 1)}, \quad n = 0, \pm 1, \pm 2, \dots, \quad (2.12)$$

which, for  $q \neq 1$ , can be rewritten as  $\frac{2}{1 - q_n} = \frac{2}{1 - q} + n$ . Note that  $q_n \equiv 1$  for all  $n = 0, \pm 1, \pm 2, \dots$ , if  $q = 1$  and  $\lim_{n \rightarrow \pm \infty} z_n(q) = 1$  for all  $q \neq 1$ . It follows from (2.12) that  $q_n > 1$  for all  $n < 2/(q - 1), q \in (1, 3)$ . Moreover, obviously,  $2/(q - 1) > 1$ , if  $q \in (1, 3)$ , which implies  $q_1 > 1$ . Generalizing what has been said, we can conclude that the condition  $q_n > 1$  guarantees  $q_k > 1, k = n - 1, n, n + 1$ , for three consequent members of the sequence (2.12).

Let us note also that the definition of the sequence  $q_n$  can be given through the following series of mappings.

**Definition 2.11.**

$$\dots \xrightarrow{z} q_{-2} \xrightarrow{z} q_{-1} \xrightarrow{z} q_0 = q \xrightarrow{z} q_1 \xrightarrow{z} q_2 \xrightarrow{z} \dots \quad (2.13)$$

$$\dots \xleftarrow{z^{-1}} q_{-2} \xleftarrow{z^{-1}} q_{-1} \xleftarrow{z^{-1}} q_0 = q \xleftarrow{z^{-1}} q_1 \xleftarrow{z^{-1}} q_2 \xleftarrow{z^{-1}} \dots \quad (2.14)$$

Furthermore, we set, for  $k = 1, 2, \dots$  and  $n = 0, \pm 1, \dots$ ,

$$F_{q_n}^k = F_{q_{n+k-1}} \circ \dots \circ F_{q_n},$$

and

$$F_{q_n}^{-k} = F_{q_{n-k}}^{-1} \circ \dots \circ F_{q_{n-1}}^{-1}.$$

Additionally, for  $k = 0$  we let  $F_q^0[f] = f$ . Summarizing the above mentioned relationships related to  $z_n(q)$ , we obtain the following assertions.

**Lemma 2.12.** *There hold the following duality relations*

$$q_{n-1} + \frac{1}{q_{n+1}} = 2, \quad n \in \mathcal{Z}. \tag{2.15}$$

*Proof.* Making use the properties (2.9), we obtain

$$q_{n-1} = z^{-1}(q_n) = 2 - z(2 - q_n) = 2 - \frac{1}{z(q_n)} = 2 - \frac{1}{q_{n+1}}. \quad \square$$

**Lemma 2.13.** *The following mappings hold:*

$$F_{q_n}^k : \mathcal{G}_{q_n} \rightarrow \mathcal{G}_{q_{k+n}}, \quad k, n \in \mathcal{Z};$$

$$\lim_{k \rightarrow -\infty} F_{q_n}^k \mathcal{G}_{q_n} = \mathcal{G}$$

where  $\mathcal{G}$  is the set of classical Gaussians.

**Lemma 2.14.** *The following mappings hold:*

$$\dots \xrightarrow{F_{q-3}} \mathcal{G}_{q-2} \xrightarrow{F_{q-2}} \mathcal{G}_{q-1} \xrightarrow{F_{q-2}} \mathcal{G}_q \xrightarrow{F_q} \mathcal{G}_{q_1} \xrightarrow{F_{q_1}} \mathcal{G}_{q_2} \xrightarrow{F_{q_2}} \dots \tag{2.16}$$

$$\dots \xleftarrow{F_{q-3}^{-1}} \mathcal{G}_{q-2} \xleftarrow{F_{q-2}^{-1}} \mathcal{G}_{q-1} \xleftarrow{F_{q-1}^{-1}} \mathcal{G}_q \xleftarrow{F_q^{-1}} \mathcal{G}_{q_1} \xleftarrow{F_{q_1}^{-1}} \mathcal{G}_{q_2} \xleftarrow{F_{q_2}^{-1}} \dots \tag{2.17}$$

Note that right sides of sequences in (2.16) and (2.17) cut off for  $n \geq 2/(q - 1)$ .

### 3. $q$ -Generalization of the Central Limit Theorem

#### 3.1. $q$ -independent random variables

In this section we establish a  $q$ -generalization of the classical CLT (see, e.g. [22, 23]) for independent identically distributed random variables with a finite variance. First we introduce some notions necessary to formulate the corresponding results. Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $X$  be a random variable defined on it with a density  $f \in L_q(R)$ ,  $\nu_q(f) = \|f\|_{L_q}^q = \int_{-\infty}^{\infty} [f(x)]^q dx < \infty$ . Introduce the density

$$f_q(x) = \frac{[f(x)]^q}{\nu_q(f)},$$

which is commonly referred to as the *escort density* [30]. Further, introduce for  $X$  the notions of  $q$ -mean,  $\mu_q = \mu_q(X) = \int_{-\infty}^{\infty} x f_q(x) dx$ , and  $q$ -variance

$\sigma_q^2 = \sigma_q^2(X - \mu_q) = \int_{-\infty}^{\infty} (x - \mu_q)^2 f_q(x) dx$ , subject to the integrals used in these definitions to converge.

The formulas below can be verified directly.

**Lemma 3.1.** *The following formulas hold true*

1.  $\mu_q(aX) = a\mu_q(X)$ ;
2.  $\mu_q(X - \mu_q(X)) = 0$ ;
3.  $\sigma_q^2(aX) = a^2\sigma_q^2(X)$ .

Further, we introduce the notions of  $q$ -independence and  $q$ -convergence.

**Definition 3.2.** Two random variables  $X_1$  and  $Y_1$  are said to be

1.  $q$ -independent of first type if

$$F_q[X + Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi), \tag{3.1}$$

2.  $q$ -independent of second type if

$$F_{q-1}[X + Y](\xi) = F_{q-1}[X](\xi) \otimes_q F_{q-1}[Y](\xi), \quad q = z(q-1), \tag{3.2}$$

3.  $q$ -independent of third type if

$$F_{q-1}[X + Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi), \quad q = z(q-1), \tag{3.3}$$

where  $X = X_1 - \mu_q(X_1)$ ,  $Y = Y_1 - \mu_q(Y_1)$ .

All three types of  $q$ -independence generalize the classic notion of independence. Namely, for  $q = 1$  the conditions (3.1)-(3.3) turn into the well known relation

$$F[f_X * f_Y] = F[f_X] \cdot F[f_Y]$$

between the convolution (noted  $*$ ) of two densities and the multiplication of their (classical) Fourier images, and holds for independent  $X$  and  $Y$ . If  $q \neq 1$ , then  $q$ -independence of a certain type describes a specific correlation. The relations (3.1)-(3.3) can be rewritten in terms of densities. Let  $f_X$  and  $f_Y$  be densities of  $X$  and  $Y$  respectively, and let  $f_{X+Y}$  be the density of  $X + Y$ . Then, for instance, the  $q$ -independence of second type takes the form

$$\int_{-\infty}^{\infty} e^{ix\xi} \otimes_{q-1} f_{X+Y}(x) dx = F_{q-1}[f_X](\xi) \otimes_q F_{q-1}[f_Y](\xi). \tag{3.4}$$

Consider an example of  $q$ -independence of second type. Let random variables  $X$  and  $Y$  have  $q$ -Gaussian densities  $G_{q-1}(\beta_1, x)$  and  $G_{q-1}(\beta_2, x)$  respectively. Denote  $\gamma_j = \frac{3-q-1}{8\beta_j^{2-q-1} C_{q-1}^{2(q-1-1)}}$ ,  $j = 1, 2$ . If  $X + Y$  is distributed

according to the density  $G_{q-1}(\delta, x)$ , where  $\delta = \left( \frac{3-q-1}{8(\gamma_1+\gamma_2)C_{q-1}^{2(q-1-1)}} \right)^{\frac{1}{2-q-1}}$ , then (3.4) is satisfied. Hence,  $X$  and  $Y$  are  $q$ -independent of second type.

The reader can easily modify the definition of  $q$ -independence to the more general case of  $q_k$ -independence. For example, relation (3.2) in the case of  $q_k$ -independence takes the form

$$F_{q_{k-1}}[X + Y](\xi) = F_{q_{k-1}}[X](\xi) \otimes_{q_k} F_{q_{k-1}}[Y](\xi), \quad q_k = z(q_{k-1}). \quad (3.5)$$

**Definition 3.3.** Let  $X_1, X_2, \dots, X_N, \dots$  be a sequence of identically distributed random variables. Denote  $Y_N = X_1 + \dots + X_N$ . By definition,  $X_N, N = 1, 2, \dots$  is said to be  $q_k$ -independent (or  $q_k$ -i.i.d.) of first type if, for all  $N = 2, 3, \dots$ , the relations

$$\begin{aligned} F_{q_k}[Y_N - N\mu_k](\xi) &= F_{q_k}[X_1 - \mu_k](\xi) \otimes_{q_k} \dots \otimes_{q_k} F_{q_k}[X_N - \mu_k](\xi), \\ \mu_k &= \mu_{q_k}(X_1), \end{aligned} \quad (3.6)$$

hold.

Analogously,  $q_k$ -independence of second and third types can be defined for sequences of identically distributed random variables.

*Remark 3.4.* For  $k = -1$  it follows from this definition the  $q$ -independence of a sequence of random variables, namely

$$\begin{aligned} F_{q-1}[Y_N - N\mu_{-1}](\xi) &= F_{q-1}[X_1 - \mu_{-1}](\xi) \otimes_q \dots \otimes_q F_{q-1}[X_N - \mu_{-1}](\xi), \\ N &= 2, 3, \dots \end{aligned} \quad (3.7)$$

*Example.* Assume that  $X_N, N = 1, 2, \dots$ , is the sequence of identically distributed random variables with the associated density  $G_{q-1}(\beta, x)$ . Further, assume the sums  $X_1 + \dots + X_N, N = 2, 3, \dots$ , are distributed according to the density  $G_{q-1}(\alpha, x)$ , where  $\alpha = N^{-\frac{1}{2-q-1}}\beta$ . Then the sequence  $X_N$  satisfies the relation (3.7) for all  $N = 2, 3, \dots$ , thus being a  $q$ -independent identically distributed sequence of random variables.

**Definition 3.5.** A sequence of random variables  $X_N, N = 1, 2, \dots$ , is said to be  $q$ -convergent to a random variable  $X_\infty$  if  $\lim_{N \rightarrow \infty} F_q[X_N](\xi) = F_q[X_\infty](\xi)$  locally uniformly in  $\xi$ .

Evidently, this definition is equivalent to the weak convergence of random variables if  $q = 1$ . For  $q \neq 1$  denote by  $W_q$  the set of continuous functions  $\phi$  satisfying the condition  $|\phi(x)| \leq C(1 + |x|)^{-\frac{q}{q-1}}, x \in R$ .

**Definition 3.6.** A sequence of random variables  $X_N$  with the density  $f_N$  is called weakly  $q$ -convergent to a random variable  $X_\infty$  with the density  $f$  if  $\int_{\mathbb{R}} f_N(x) dm_q \rightarrow \int_{\mathbb{R}^d} f(x) dm_q$  for arbitrary measure  $m_q$  defined as  $dm_q(x) = \phi_q(x) dx$ , where  $\phi_q \in W_q$ .

The  $q$ -weak convergence is equivalent to the  $q$ -convergence [25] (details will be discussed in a separate paper).

We will study limits of sums

$$Z_N = \frac{1}{D_N(q)} (X_1 + \dots + X_N - N\mu_q), \quad N = 1, 2, \dots$$

where  $D_N(q) = (\sqrt{N\nu_{2q-1}\sigma_{2q-1}})^{\frac{1}{2-q}}$ ,  $N = 1, 2, \dots$ , in the sense of Definition 3.5, when  $N \rightarrow \infty$ . Namely, the question we are interested in: *Is there a  $q$ -normal distribution that attracts the sequence  $Z_N$ ?* For  $q = 1$  the answer is well known and it is the content of the classical central limit theorem.

### 3.2. Main results

The formulation of a generalization of the central limit theorem consistent with nonextensive statistical mechanics depends on the type of  $q$ -independence. We prove the  $q$ -generalization of the central limit theorem under the condition of first type of  $q$ -independence.

**Theorem 1.** *Assume a sequence  $q_k, k \in \mathbb{Z}$ , is given as (2.12) with  $q_k \in [1, 2)$ . Let  $X_1, \dots, X_N, \dots$  be a sequence of  $q_k$ -independent (for a fixed  $k$ ) of first type and identically distributed random variables with a finite  $q_k$ -mean  $\mu_{q_k}$  and a finite second  $(2q_k - 1)$ -moment  $\sigma_{2q_k-1}^2$ .*

*Then  $Z_N = \frac{X_1 + \dots + X_N - N\mu_{q_k}}{D_N(q_k)}$  is  $q_k$ -convergent to a  $q_{k-1}$ -normal distribution as  $N \rightarrow \infty$ . Moreover, the corresponding attractor is  $G_{q_{k-1}}(\beta_k; x)$  with*

$$\beta_k = \left( \frac{3 - q_{k-1}}{4q_k C_{q_{k-1}}^{2q_{k-1}-2}} \right)^{\frac{1}{2-q_{k-1}}}. \tag{3.8}$$

The proof of this theorem follows from Theorem 2 proved below and Lemma 2.14. Theorem 2 represents one element ( $k = 0$ ) in the series of assertions contained in Theorem 1.

**Theorem 2.** *Assume  $1 \leq q < 2$ . Let  $X_1, \dots, X_N, \dots$  be a sequence of  $q$ -independent of first type and identically distributed random variables with a finite  $q$ -mean  $\mu_q$  and a finite second  $(2q - 1)$ -moment  $\sigma_{2q-1}^2$ .*

Then  $Z_N = \frac{X_1 + \dots + X_N - N\mu_q}{D_N(q)}$  is  $q$ -convergent to a  $q_{-1}$ -normal distribution as  $N \rightarrow \infty$ . The corresponding  $q_{-1}$ -Gaussian is  $G_{q_{-1}}(\beta; x)$ , with

$$\beta = \left( \frac{3 - q_{-1}}{4qC_{q_{-1}}^{2q_{-1}-2}} \right)^{\frac{1}{2-q_{-1}}}.$$

*Proof.* Let  $f$  be the density associated with  $X_1 - \mu_q$ . First we evaluate  $F_q(X_1 - \mu_q) = F_q(f(x))$ . Using Lemma 2.2 we have

$$F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx = \int_{-\infty}^{\infty} f(x) e_q^{ix\xi[f(x)]^{q-1}} dx. \quad (3.9)$$

Making use of the asymptotic expansion  $e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2)$ ,  $x \rightarrow 0$ , we can rewrite the right hand side of (3.9) in the form

$$\begin{aligned} & F_q[f](\xi) \\ &= \int_{-\infty}^{\infty} f(x) \left( 1 + ix\xi[f(x)]^{q-1} - \frac{q}{2}x^2\xi^2[f(x)]^{2(q-1)} + o(x^2\xi^2[f(x)]^{2(q-1)}) \right) dx \\ &= 1 + i\xi\mu_q\nu_q - \frac{q}{2}\xi^2\sigma_{2q-1}^2\nu_{2q-1} + o(\xi^2), \quad \xi \rightarrow 0. \end{aligned} \quad (3.10)$$

In accordance with the condition of the theorem and the relation (2) in Lemma 3.1,  $\mu_q = 0$ . Denote  $Y_j = D_N(q)^{-1}(X_j - \mu_q)$ ,  $j = 1, 2, \dots$ . Then  $Z_N = Y_1 + \dots + Y_N$ . Further, it is readily seen that, for a given random variable  $X$  and real  $a > 0$ , there holds  $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$ . It follows from this relation that  $F_q(Y_1) = F_q[f]\left(\frac{\xi}{\sqrt{N\nu_{2q-1}\sigma_{2q-1}}}\right)$ . Moreover, it follows from the  $q$ -independence of  $X_1, X_2, \dots$  and the associativity of the  $q$ -product that

$$F_q[Z_N](\xi) = F_q[f]\left(\frac{\xi}{\sqrt{N\nu_{2q-1}\sigma_{2q-1}}}\right) \otimes_q \dots \otimes_q F_q[f]\left(\frac{\xi}{\sqrt{N\nu_{2q-1}\sigma_{2q-1}}}\right) \quad (N \text{ factors}). \quad (3.11)$$

Hence, making use of properties of the  $q$ -logarithm, from (3.11) we obtain

$$\begin{aligned} \ln_q F_q[Z_N](\xi) &= N \ln_q F_q[f]\left(\frac{\xi}{\sqrt{N\nu_{2q-1}\sigma_{2q-1}}}\right) = N \ln_q \left( 1 - \frac{q}{2} \frac{\xi^2}{N} + o\left(\frac{\xi^2}{N}\right) \right) \\ &= -\frac{q}{2}\xi^2 + o(1), \quad N \rightarrow \infty, \end{aligned} \quad (3.12)$$

locally uniformly by  $\xi$ .

Consequently, locally uniformly by  $\xi$ ,

$$\lim_{N \rightarrow \infty} F_q(Z_N) = e_q^{-\frac{q}{2}\xi^2}. \tag{3.13}$$

Thus,  $Z_N$  is  $q$ -convergent to the random variable  $Z$  whose  $q$ -Fourier transform is  $e_q^{-\frac{q}{2}\xi^2} \in \mathcal{G}_q$ .

In accordance with Corollary 2.6 for  $q$  and some  $\beta$  there exists a density  $G_{q_{-1}}(\beta; x)$ ,  $q = z(q_{-1})$ , such that  $F_{q_{-1}}(G_{q_{-1}}(\beta; x)) = e_q^{-(q/2)\xi^2}$ . Let us now find  $\beta$ . It follows from Corollary 2.6 (see (2.11)) that  $\beta_*(q_{-1}) = q/2$ . Solving this equation with respect to  $\beta$  we obtain

$$\beta = \left( \frac{3 - q_{-1}}{4qC_{q_{-1}}^{2(q_{-1}-1)}} \right)^{\frac{1}{2-q_{-1}}}, \tag{3.14}$$

The explicit form of the corresponding  $q_{-1}$ -Gaussian reads as

$$G_{q_{-1}}(\beta; x) = C_s^{-1} \left( \frac{\sqrt{3-s}}{2C_s^{s-1}\sqrt{z(s)}} \right)^{\frac{1}{2-s}} e_s^{-\left( \frac{3-s}{4z(s)C_s^{2(s-1)}} \right)^{\frac{1}{2-s}} x^2}, \quad s = q_{-1}. \tag{3.15}$$

□

Analogously, the  $q$ -CLT can be proved for  $q_k$ -i.i.d. of the second and third types. The formulations of the corresponding theorems are given below. The reader can readily verify their validity through comparison with the proof of Theorem 1.

**Theorem 3.** Assume a sequence  $q_k, k \in \mathcal{Z}$ , is given as (2.12) with  $q_k \in [1, 2)$ . Let  $X_1, \dots, X_N, \dots$  be a sequence of  $q_k$ -independent (for a fixed  $k$ ) of second type and identically distributed random variables with a finite  $q_{k-1}$ -mean  $\mu_{q_{k-1}}$  and a finite second  $(2q_{k-1} - 1)$ -moment  $\sigma_{2q_{k-1}-1}^2$ .

Then  $Z_N = \frac{X_1 + \dots + X_N - N\mu_{q_{k-1}}}{D_N(q_{k-1})}$  is  $q_{k-1}$ -convergent to a  $q_{k-1}$ -normal distribution as  $N \rightarrow \infty$ . The parameter  $\beta_k$  of the corresponding attractor  $G_{q_{k-1}}(\beta_k; x)$  is

$$\beta_k = \left( \frac{3 - q_{k-1}}{4q_{k-1}C_{q_{k-1}}^{2q_{k-1}-2}} \right)^{\frac{1}{2-q_{k-1}}}. \tag{3.16}$$

**Theorem 4.** Assume a sequence  $q_k, k \in \mathcal{Z}$ , is given as (2.12) with  $q_k \in [1, 2)$ . Let  $X_1, \dots, X_N, \dots$  be a sequence of  $q_k$ -independent (for a fixed  $k$ ) of third type and identically distributed random variables with a finite  $q_k$ -mean  $\mu_{q_k}$  and a finite second  $(2q_k - 1)$ -moment  $\sigma_{2q_k-1}^2$ .

Then  $Z_N = \frac{X_1 + \dots + X_N - N\mu_{q_k}}{D_N(q_k)}$  is  $q_{k-1}$ -convergent to a  $q_{k-1}$ -normal distribution as  $N \rightarrow \infty$ . Moreover, the corresponding attractor  $G_{q_{k-1}}(\beta_k; x)$  in this case is the same as in the Theorem 1 with  $\beta_k$  given in (3.8).

Obviously,  $\frac{q+1}{3-q} = 1$  if and only if  $q = 1$ . This fact yields the following corollary.

**Corollary 3.7.** *Let  $X_1, \dots, X_N, \dots$  be a given sequence of  $q_k$ -independent (of any type) random variables satisfying the corresponding conditions of Theorems 1–3. Then the attractor of  $Z_N$  is a  $q_k$ -normal distribution if and only if  $q_k = 1$ , that is, in the classic case.*

### 4. Conclusion

In the present paper we studied  $q$ -generalizations of the classic central limit theorem adapted to nonextensive statistical mechanics, depending on three types of correlation. Interrelation showing the dependence of a type of convergence to a  $q$ -Gaussian on a type of correlation is summarized in Table 1.

TABLE 1. Interrelation between the type of correlation, conditions for  $q$ -mean and  $q$ -variance, type of convergence, and parameter of the attractor

Corr. type	Conditions	Convergence	Gaussian parameter
1st type	$\mu_{q_k} < \infty,$ $\sigma_{2q_k-1}^2 < \infty$	$q_k - conv.$	$\beta_k = \left( \frac{3-q_{k-1}}{4q_k C_{q_{k-1}}^{2q_{k-1}-2}} \right)^{\frac{1}{2-q_{k-1}}}$
2nd type	$\mu_{q_{k-1}} < \infty,$ $\sigma_{2q_{k-1}-1}^2 < \infty$	$q_{k-1} - conv.$	$\beta_k = \left( \frac{3-q_{k-1}}{4q_{k-1} C_{q_{k-1}}^{2q_{k-1}-2}} \right)^{\frac{1}{2-q_{k-1}}}$
3rd type	$\mu_{q_k} < \infty,$ $\sigma_{2q_k-1}^2 < \infty$	$q_{k-1} - conv.$	$\beta_k = \left( \frac{3-q_{k-1}}{4q_k C_{q_{k-1}}^{2q_{k-1}-2}} \right)^{\frac{1}{2-q_{k-1}}}$

In all three cases the corresponding attractor is distributed according to a  $q_{k-1}$ -normal distribution, where  $q_{k-1} = z^{-1}(q_k)$ . We have noticed that  $q_k \neq q_{k-1}$  if  $q_k \neq 1$ . In the classic case both  $q_k = 1$  and  $z^{-1}(q_k) = 1$ , so that the corresponding Gaussians do not differ. So, Corollary 3.7 notes that such duality is a specific feature of the nonextensive statistical theory, which



comes from specific correlations of  $X_j$ , which cause in turn nonextensivity of the phenomenon under study.

Now let us briefly discuss the scaling rate, important notion in diffusion theory. We recall that the standard Gaussian evolved in time can be calculated

$$G(t, x) = F^{-1}[e^{-t\xi^2}] = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}, \quad t > 0.$$

It follows immediately that the mean squared displacement is related to time with the scaling rate  $x^2 \sim t^\delta$  with  $\delta = 1$ . Applying the same technique in the case of the  $q$ -theory, in all three types of correlations, using (3.15), Lemma 2.12 and expression for  $\beta_k$  (see fourth column of Table 1), we obtain  $\delta = 1/(2 - q_{k-1}) = q_{k+1}$ . Hence, three consequent members of  $q_k = \frac{2q+k(1-q)}{2+k(1-q)}$ ,  $q \in [1, 2)$ , namely the triplet  $(q_{k-1}, q_k, q_{k+1})$ , play an important role in the description of a nonextensive phenomenon. Namely, if a correlation is given by  $q_k$ , then the corresponding attractor is a  $q_{k-1}$ -Gaussian and, in turn, the scaling rate is equal to  $q_{k+1}$ .

Finally, we would like to address connections of the theorems that we have proved with two known results. We have seen that the classical central limit theorem may in principle be generalized in various manners, each of them referring to correlations of specific kinds. In [4], an example of correlated model was discussed numerically. The correlations were introduced, in a scale-invariant manner, through a  $q$ -product in the space of the joint probabilities of  $N$  binary variables, with  $0 \leq q \leq 1$ <sup>3</sup>. It was numerically shown that the attractors are very close (although not exactly [27]) to (double-branched)  $Q$ -Gaussians, with  $Q = 2 - \frac{1}{q} \in (-\infty, 1]$ , and that the model is superdiffusive [28] with  $1 \leq \delta \leq 2$ ). The relation  $Q = 2 - \frac{1}{q}$  corresponds to the particular case  $k = -1$  of the present Theorem 1. It comes from Lemma 2.12, with  $q_{-2} = 2 - \frac{1}{q_0} = 2 - \frac{1}{q}$ , which holds when  $k = -1$ . It should be noted the following connection between these two models, related to the behavior of the scaling rate  $\delta$ . In the model introduced in [4], superdiffusion occurs with  $\delta$  monotonically decreasing from 2 to 1 when  $q$  increases from 0 to 1 [28]. In our  $k = -1$  model we have  $\delta = q \in [1, 3)$ , that is, monotonically increasing from 1 to 3, when  $q_{-1}$  increases from 1 to 2. The scaling rates in these two models behave as a sort of continuation of each other, through  $q = 1$ , on the interval  $q \in [0, 3)$ , and in both models we observe only superdiffusion.

<sup>3</sup>Even though we are analyzing here the case  $q \geq 1$ , it is useful to compare our model with the one in [4].

Another example is suggested by the exact stable solutions of a nonlinear Fokker-Planck equation in [6]. The correlations<sup>4</sup> are introduced through a  $q = 2 - Q$  exponent in the spatial member of the equation (the second derivative term). The solutions are  $Q$ -Gaussians with  $Q \in (-\infty, 3)$ , and  $\delta = 2/(3 - Q) \in [0, \infty]$ , hence both superdiffusion and subdiffusion can exist in addition to normal diffusion. This model is particularly interesting because the scaling  $\delta = 2/(3 - Q)$  was conjectured in [31], and it was verified in various experimental and computational studies [32, 33, 34, 35].

In the particular case of Theorem 1,  $k = 1$ , we have  $\delta = q_2 = 1/(2 - q)$ . This result coincides with that of the nonlinear Fokker-Planck equation mentioned above. Indeed, in our theorem ( $k = 1$ ) we require the finiteness of  $(2q - 1)$ -variance. Denoting  $2q - 1 = Q$ , we get  $\delta = 1/(2 - q) = 2/(3 - Q)$ . Notice, however, that this example differs from the nonlinear Fokker-Planck above. Indeed, although we do obtain, from the finiteness of the second moment, the same expression for  $\delta$ , the attractor is not a  $Q$ -Gaussian, but rather a  $\frac{Q+1}{2}$ -Gaussian.

Summarizing, the present Theorems 1–4 suggest a quite general and rich structure at the basis of nonextensive statistical mechanics. Moreover, they recover, as particular instances, central relations emerging in the above two examples. The structure we have presently shown might pave a deep understanding of the so-called  $q$ -triplet  $(q_{sen}, q_{rel}, q_{stat})$ , where *sen*, *rel* and *stat* respectively stand for *sensitivity to the initial conditions*, *relaxation*, and *stationary state* [36, 37, 38] in nonextensive statistics. This remains however as a challenge at the present stage. Another open question – very relevant in what concerns physical applications – refers to whether the  $q$ -independence addressed here reflects a sort of asymptotic scale-invariance as  $N$  increases.

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<sup>4</sup>The correlation defined in [6] is different from the  $q$ -independence of types 1-3 introduced in this paper.

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Sabir Umarov  
Department of Mathematics  
Tufts University  
Medford, MA 02155  
USA  
e-mail: [Sabir.Umarov@tufts.edu](mailto:Sabir.Umarov@tufts.edu)

Constantino Tsallis  
Centro Brasileiro de Pesquisas Fisicas  
Xavier Sigaud 150  
22290-180 Rio de Janeiro-RJ  
Brazil

and

Santa Fe Institute  
1399 Hyde Park Road  
Santa Fe, NM 87501  
USA  
e-mail: [tsallis@cbpf.br](mailto:tsallis@cbpf.br)

Stanly Steinberg  
Department of Mathematics and Statistics  
University of New Mexico  
Albuquerque, NM 87131  
USA  
e-mail: [stanly@math.unm.edu](mailto:stanly@math.unm.edu)

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