Generalization of Shannon's theorem for Tsallis entropy

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By using the assumptions that the entropy must (i) be a continuous function of the probabilities $\{p_i\}(p_i \in (0,1) \forall i)$, only; (ii) be a monotonic increasing function of the number of states W, in the case of equiprobability; (iii) satisfy the pseudoadditivity relation $S_q(A+B)/k = S_q(A)/k + S_q(B)/k + (1-q)S_q(A)S_q(B)/k^2$ (A and B being two independent systems, $q \in \Re$ and k a positive constant), and (iv) satisfy the relation $S_q(\{p_i\}) = S_q(p_L, p_M) + p_L{}^q S_q(\{p_i/p_L\}) + p_M{}^q S_q(\{p_i/p_M\})$, where $p_L + p_M = 1 (p_L = \Sigma_{i=1}^{W_L} p_i$ and $p_M = \Sigma_{i=W_L+1}^{W} p_i)$, we prove, along Shannon's lines, that the unique function that satisfies all these properties is the generalized Tsallis entropy $S_q = k(1 - \Sigma_{i=1}^{W} p_i^q)/(q-1)$. © *1997 American Institute of Physics.* [S0022-2488(97)03607-4]

I. INTRODUCTION

Nonextensive physical systems are being intensively studied nowadays. A recently introduced formalism, namely Tsallis statistics,[1] has been proposed in order to cover many of such anomalous systems. Indeed, it has been successfully applied to Lévy-type[2–4] and correlated-type[5–8] anomalous diffusions, turbulence in electron plasmas,[9] the solar neutrino problem,[10] quantum groups,[11–13] nonlinear dynamical systems,[14] cosmology,[15] linear response theory,[16] as well as to optimization technics.[17–23]

This thermo-statistical formalism is based upon the so-called Tsallis entropy formula

$$S_q = k\,\frac{1-\Sigma_{i=1}^{W} p_i^q}{q-1} \quad (q \in \Re), \tag{1}$$

where k is a positive constant (which we shall from now on take equal to 1), q is a real number, W is the total number of microscopic configurations, and $\{p_i\}$ is the set of associated probabilities $(\Sigma_{i=1}^{W} p_i = 1)$. It is easily seen that in the limit $q \to 1$, one recovers the well-known Boltzmann–Gibbs–Shannon formula[21]

$$S = -\sum_{i=1}^{W} p_i \ln p_i, \tag{2}$$

which successfully accounts for extensive problems.

It is known that S_q satisfies the following conditions:

(i) S_q is, for $0 < p_i < 1$, a continuous function of $\{p_i\}$, only.

(ii) For a given set of W equiprobable states, i.e., $p_i = 1/W$, S_q is a monotonic increasing function of W, namely $S_q = (W^{1-q} - 1)/(1-q)$.

(iii) For two independent systems A and B, the generalized entropy of the composed system $A+B$ satisfies the pseudoadditivity relation (see, for instance, Ref. 11):

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B). \tag{3}$$

(iv) With

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$$W = W_L + W_M, \tag{4}$$

$$p_L = \sum_{i=1}^{W_L} p_i \quad (W_L \text{ terms}), \tag{5}$$

$$p_M = \sum_{i=W_L+1}^{W} p_i \quad (W_M \text{ terms}), \tag{6}$$

$$(\text{hence, } \; p_L + p_M = 1) \tag{7}$$

we have:[1,20]

$$S_q(\{p_i\}) = S_q(p_L, p_M) + p_L{}^q S_q\left(\left\{\frac{p_i}{p_L}\right\}\right) + p_M{}^q S_q\left(\left\{\frac{p_i}{p_M}\right\}\right). \tag{8}$$

In this letter we show that the *unique* function that simultaneously satisfies all these properties is the Tsallis generalized entropy formula (1). By so doing, we are generalizing, for the case of nonextensive systems, the famous Shannon's theorem.[24]

II. PROOF

Let us decompose a choice from s^m equally likely possibilities into a series of m choices with s equally likely possibilities each. It is straightforward to show that using condition (iii) one gets:

$$S_q(s^m) = \frac{[1+(1-q)S_q(s)]^m - 1}{1-q}. \tag{9}$$

This expression is the generalization for $q \in \mathfrak{R}$ of the extensivity condition of the Boltzmann–Gibbs–Shannon entropy, and, in the limit $q \to 1$, yields the well-known result[24]

$$S_1(s^m) = m S_1(s). \tag{10}$$

For a large enough m and s it is always possible to find a pair of integer numbers (t,n) such that

$$s^m \leqslant t^n \leqslant s^{m+1}. \tag{11}$$

Now, from condition (ii), we have, for all values of q:

$$S_q(s^m) \leqslant S_q(t^n) \leqslant S_q(s^{m+1}) \tag{12}$$

so, using Eq. (9) for $q<1$,

$$[1+(1-q)S_q(s)]^m \leqslant [1+(1-q)S_q(t)]^n \leqslant [1+(1-q)S_q(s)]^{m+1}. \tag{13}$$

Taking the logarithm of this inequality we get

$$\frac{m}{n} \leqslant \frac{\ln[1+(1-q)S_q(t)]}{\ln[1+(1-q)S_q(s)]} \leqslant \frac{m}{n} + \frac{1}{n} \tag{14}$$

or equivalently,

$$\left|\frac{m}{n} - \frac{\ln[1+(1-q)S_q(t)]}{\ln[1+(1-q)S_q(s)]}\right| \leqslant \epsilon \equiv \frac{1}{n}. \tag{15}$$

Now, from Eq. (11), we have

$$\frac{m}{n} \leqslant \frac{\ln t}{\ln s} \leqslant \frac{m}{n} + \frac{1}{n} \tag{16}$$

and, as before,

$$\left| \frac{m}{n} - \frac{\ln t}{\ln s} \right| \leqslant \epsilon \equiv \frac{1}{n}. \tag{17}$$

Combining Eqs. (15) and (17) there comes

$$\left| \frac{\ln t}{\ln s} - \frac{\ln[1+(1-q)S_q(t)]}{\ln[1+(1-q)S_q(s)]} \right| \leqslant 2\epsilon. \tag{18}$$

If we now allow $\epsilon \to 0$ we get

$$\frac{\ln[1+(1-q)S_q(s)]}{\ln s} = \frac{\ln[1+(1-q)S_q(t)]}{\ln t} = p(q), \tag{19}$$

where $p(q)$ is a quantity which at most can depend on q.
So we get the functional form of $S_q(t)$ given by:

$$S_q(t) = \frac{t^{p(q)}-1}{1-q}. \tag{20}$$

Let us now consider a choice from W partitions, each one with probability

$$p_i = \frac{n_i}{\Sigma_{i=1}^{W} n_1}, \tag{21}$$

where n_i is the number of possibilities in the ith partition, each one with equal probability. Using condition (iv) expressed in Eq. (8) we get

$$S_q\left(\left\{ \frac{1}{\Sigma_{i=1}^{W} n_1} \right\} \right) = S_q(p_1, p_2, \ldots, p_W) + \sum_{i=1}^{W} p_i^q S_q\left(\left\{ \frac{1}{n_i} \right\} \right) \tag{22}$$

or, using the functional form of Eq. (20),

$$\frac{(\Sigma_{i=1}^{W} n_i)^p - 1}{1-q} = S_q(p_1, p_2, \ldots, p_W) + \sum_{i=1}^{W} p_i^q \left(\frac{n_i^q - 1}{1-q} \right) \tag{23}$$

so

$$S_q\left(p_1, p_2, \ldots, p_W \right) = \frac{1}{1-q} \left\{ \left(\sum_{i=1}^{W} n_i \right)^p - 1 + \sum_{i=1}^{W} p_i^q - \sum_{i=1}^{W} p_i^q n_i^p \right\} \tag{24}$$

but

$$n_i^p = p_i^p \left(\sum_{i=1}^{W} n_i \right)^p. \tag{25}$$

Thus, substituting into Eq. (24), we see that, in order to satisfy condition (i) so that $S_q(\{p_i\})$ must depend only on $\{p_i\}$, one must have

$$p=1-q. \tag{26}$$

Therefore

$$S_q(\{p_i\})=\frac{1-\Sigma_{i=1}^{W}p_i^q}{q-1} \tag{27}$$

and, for t equiprobable choices,

$$S_q(t)=\frac{t^{1-q}-1}{q-1}. \tag{28}$$

In this way we have generalized, for the Tsallis entropy, Shannon's remarkable theorem.

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