Nonextensivity and Multifractality in Low-Dimensional Dissipative Systems

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(Received 22 September 1997)

Power-law sensitivity to the initial conditions at the edge of chaos provides a natural relation between the scaling properties of the dynamics attractor and its degree of nonextensivity within the generalized statistics recently introduced by one of the authors (C.T.) and characterized by the entropic index q. We show that general scaling arguments imply that $1/(1-q) = 1/\alpha_{\min} - 1/\alpha_{\max}$, where $\alpha_{\min}$ and $\alpha_{\max}$ are the extremes of the multifractal singularity spectrum $f(\alpha)$ of the attractor. This relation is numerically verified in standard $D = 1$ dissipative maps. [S0031-9007(97)04926-0]

PACS numbers: 05.45.+b, 05.20.–y, 05.70.Ce

Nonextensivity is inherent in systems where long-range interactions or spatiotemporal complexity are present. Long-range forces are found at astrophysical as well as nanometric scales. Spatiotemporal complexity, a term introduced to describe the presence of long-range spatial and temporal correlations, is found in equilibrium statistical mechanics to emerge at critical points for second order phase transitions. Further, the concept of self-organized criticality has been recently introduced to describe driven systems which naturally evolve to a dynamical attractor poised at criticality [1]. Self-organized criticality is conjectured to be in the origin of fractal structures, noise with a $1/f$ power spectrum, anomalous diffusion, Lévy flights, and punctuated equilibrium behavior [2], which are signatures of the nonextensive character of the dynamics attractor.

The proper statistical treatment of nonextensive systems seems to require a generalization of the Boltzmann-Gibbs-Shannon prescription based in the standard, extensive entropy $S = -\sum_i p_i \ln p_i$ (in units of Boltzmann constant). Inspired by the scaling properties of multifractals, one of us [3] has proposed a generalized nonextensive form of entropy,

$$S_q = \frac{1 - \sum_i p_i^q}{q - 1}, \qquad \left(\sum_i p_i = 1; q \in \mathcal{R}\right), \tag{1}$$

which recovers the usual entropy form in the limit of $q \to 1$. The entropic index q controls the degree of nonextensivity reflected in the pseudoadditivity entropy rule $S_q(A + B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$, where A and B are two *independent* systems in the sense that the probabilities of $A + B$ *factorize* into those of A and of B. Let us also mention that this generalized entropy is related to the well-known Renyi entropy (which, in contrast with the present one, is generically extensive and has no definite concavity). A wealth of studies has been developed in the past few years showing that the above nonextensive thermostatistical prescription retains much of the formal structure of the standard theory such as the Legendre thermodynamic structure, H theorem, Onsager reciprocity theorem, Kramers and Wannier relations, and thermodynamic stability, among others [4]. Further, it has been applied to a series of nonextensive systems such as stellar polytropes [5], ferrofluids [6], two-dimensional plasma turbulence [7], anomalous diffusion and Lévy flights [8], cosmology [9], peculiar velocities of galaxies [10], and inverse bremsstrahlung in plasma [11], among others [12].

In spite of these variety of applications of nonextensive thermostatistics, a full and general understanding of the *precise* relation between the entropic index q and the underlying microscopic dynamics was still lacking. It has been conjectured that the generalized thermostatistics is a natural frame for studying fractally structured systems [12], and simple relations were found between q and the characteristic exponents of anomalous diffusion and Lévy flights distributions [8]. Furthermore, recent works have shown that the entropic index q has a monotonic dependence on the fractal dimension d_f of the dynamical chaotic attractor of dissipative nonlinear systems [13].

The purpose of this paper is to develop the precise connection between the nonextensivity parameter q and the scaling properties of the critical attractor of nonlinear dynamical systems. Particularly, a prototype complex dynamical state will be taken to be the onset of chaos of nonlinear low-dimensional maps. We will show that the *power-law* sensitivity to initial conditions at the edge of chaos provides a natural link between the entropic index q and the attractor's multifractal singularity spectrum. For the sake of simplicity, we will concentrate our attention to the simple case of one-dimensional nonlinear dynamical systems. One of their most prominent features is related to their sensitivity to initial conditions. In order to quantify this aspect, Kolmogorov and Sinai's definition of the rate at which the amount of information about the initial conditions varies can be seen as

$$K_1 \equiv \lim_{\tau \to 0} \lim_{l \to 0} \lim_{N \to \infty} \frac{1}{N\tau} [S_1(N) - S_1(0)], \tag{2}$$

where τ is a characteristic time step (in fact, $\tau \to 0$

for differential equations; $\tau = 1$ for discrete maps) and $S_1(0)$ and $S_1(N)$ stand, respectively, for the entropies evaluated at the times $t = 0$ and $t = N\tau$. The entropy can be evaluated by considering an ensemble of identical copies of the system and defining p_i as the fractional number of copies that are in the ith cell (of size l) of the phase space. If one uses the extensive Boltzmann-Gibbs-Shannon entropy form $S = S_1 = -\sum_{i=1}^{W} p_i \ln p_i$, where W is the number of configurations at time t, the Kolmogorov-Sinai entropy results in

$$K_1 = \lim_{\tau\to 0} \lim_{l\to 0} \lim_{N\to\infty} \frac{1}{N\tau} \ln W(N)/W(0), \qquad (3)$$

where equiprobability ($p_i = 1/W$) was assumed. Notice that the above expression implies in an exponential sensitivity to initial conditions $W(N) = W(0)e^{K_1 N\tau}$. K_1 plays (consistently with Pesin's equality) the role of the Liapunov exponent λ_1 which characterizes the exponential deviation of two initially nearby paths $\xi(t) \equiv \lim_{\Delta x(0)\to 0} \Delta x(t)/\Delta x(0) = e^{\lambda_1 t}$ [$\xi(t)$ is the solution of $d\xi/dt = \lambda_1 \xi$]. When $\lambda_1 < 0$ ($\lambda_1 > 0$) the system is said to be *strongly insensitive* (*strongly sensitive*) to the initial conditions. The marginal case of $\lambda_1 = 0$ occurs at the period-doubling and tangent bifurcation points, as well as at the threshold to chaos. The failure of the above scheme in distinguishing the sensitivity to initial conditions at these special points is related to the nonextensive (fractal-like) structure of their dynamical attractors.

Recently it was argued that, within the generalized nonextensive entropy of Eq. (1), the sensitivity to initial conditions of one-dimensional nonlinear maps becomes expressed as [13]

$$\xi(t) = [1 + (1 - q)\lambda_q t]^{1/(1-q)}, \qquad (4)$$

which is the solution of $d\xi/dt = \lambda_q \xi^q$. The above expression recovers the usual exponential sensitivity in the limit of $q \to 1$ (extensive statistics). Further, it implies a power-law sensitivity when nonextensivity takes place ($q \neq 1$). Previous numerical calculations have shown that the period-doubling and tangent bifurcations exhibit *weak insensitivity* ($q > 1$) to initial conditions [13]. At the onset of chaos, *weak sensitivity* ($q < 1$) shows up and the value of q was numerically verified to be closely related to the fractal dimension d_f of the dynamical attractor [13]. In what follows, we will use scaling arguments to analytically express the entropic index q as a function of the fractal scaling properties of the attractor.

Actually, the scaling behavior of the critical attractor is richer and more complex than is the case in usual critical phenomena. It is necessary to introduce a multifractal formalism in order to reveal its complete scaling behavior [14]. A central quantity in this formalism is the partition function $\chi_{\bar q}(N) = \sum_{i=1}^{N} p_i^{\bar q}$, where p_i represents the probability (integrated measure) on the ith box among the N boxes of the measure (we use $\bar q$ instead of the standard notation q in order to avoid confusion with the entropic index q). In the $N \to \infty$ limit, the contribution to $\chi_{\bar q}(N) \propto N^{-\tau(\bar q)}$, with a given $\bar q$, comes from a subset of all possible boxes whose number scales as $N_{\bar q} \propto N^{f(\bar q)}$, where $f(\bar q)$ is the fractal dimension of the subset. The content on each contributing box is roughly constant and scales as $P_{\bar q} \propto N^{-\alpha(\bar q)}$. These exponents are all related by a Legendre transformation $\tau(\bar q) = \bar q \alpha(\bar q) - f(\bar q)$, and the multifractal object is characterized by the continuous function $f(\alpha)$. This formalism has been widely used to characterize some important objects arising in nonlinear dynamical systems [14,15]. The α values at the end points of the $f(\alpha)$ curve are the singularity strength associated with the regions in the set where the measure is most concentrated [$\alpha_{\min} = \alpha(\bar q = +\infty)$] and most rarefied [$\alpha_{\max} = \alpha(\bar q = -\infty)$].

The scaling properties of the most rarefied and most concentrated regions of multifractal dynamical attractors can be used to estimate the power-law divergence of nearby orbits. Consider the set of points in the attractor generated after a large number B of time steps ($p_i = 1/B$ is therefore the measure contained in each box). The most concentrated and most rarefied regions in the attractor are partitioned, respectively, in boxes of typical sizes $l_{+\infty}$ and $l_{-\infty}$. From these, one may determine the end points of the singularity spectrum as $\alpha_{\min} = \ln p_i/\ln l_{+\infty}$ (hence $l_{+\infty} \propto B^{-1/\alpha_{\min}}$) and $\alpha_{\max} = \ln p_i/\ln l_{-\infty}$ (hence $l_{-\infty} \propto B^{-1/\alpha_{\max}}$). Further, the smallest splitting between two nearby orbits, which is of the order of $l_{+\infty}$, becomes, at most, a splitting of the order of $l_{-\infty}$. With these scaling relations, Eq. (4) reads $l_{-\infty}/l_{+\infty} \propto B^{1/(1-q)}$, which implies a precise relation between the entropic index q and the extremes of $f(\alpha)$:

$$\frac{1}{1-q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}}. \qquad (5)$$

The above expression is the main result of the present work. It asserts that, once the scaling properties of the dynamical attractor are known, one can precisely infer the proper nonextensive statistics that must be used. Let us illustrate the above result using as prototype multifractal objects the critical attractor of one-dimensional dissipative maps. As has been shown by Feigenbaum, with $B = b^n$ cycle elements on the attractor (b stands for a natural scale for the partitions), the most rarefied and most concentrated elements scale, respectively, as $l_{-\infty} \sim \alpha_F^{-n}$ and $l_{+\infty} \sim \alpha_F^{-zn}$, where α_F is the Feigenbaum universal scaling factor and z represents the nonlinearity (inflexion) of the map at the vicinity of its extremal point [16]. Since the measures there are simply $p_{-\infty} = p_{+\infty} = p_i = b^{-n}$, these end points are, respectively,

$$\alpha_{\max} = \frac{\ln b}{\ln \alpha_F}, \qquad \alpha_{\min} = \frac{\ln b}{z \ln \alpha_F}. \qquad (6)$$

Therefore, the entropic index q can be put forth as a function of the Feigenbaum scaling factor as

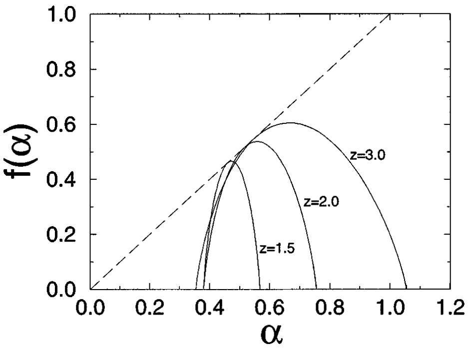

FIG. 1. Multifractal singularity spectra of the critical attractor of generalized logistic maps with $z = 1.5$, 2.0, and 3.0 as numerically obtained following the prescription in Ref. [14].

$$\frac{1}{1-q} = (z-1)\frac{\ln \alpha_F}{\ln b}. \quad (7)$$

In order to determine the singularity spectrum of the critical attractor, we implemented the algorithm proposed by Halsey *et al.* [14]. We consider a family of generalized logistic maps $x_{t+1} = 1 - a|x_t|^z$ $(1 < z < \infty; 0 < a \le 2; -1 \le x_t \le 1)$, which exhibits a period-doubling cascade accumulating at $a_c(z)$ ($b = 2$ is therefore the natural scale for the partitions). Here z is precisely the inflexion of the map at the vicinity of its extremal $\bar{x} = 0$. Typical multifractal singularity spectra are shown in Fig. 1. From their end points we can estimate the z dependence of the universal scaling factor $\alpha_F(z)$. The values obtained are shown in Fig. 2, together with known asymptotics [17] and the parametric dependence of $\alpha_{\min}$ and $\alpha_{\max}$ on the fractal dimension d_f (see inset). The entropic index q was *independently* obtained (numerically) from the plots of $\ln \xi(t) = \sum_{t=1}^{N} \ln[az|x_t|^{z-1}]$ versus $\ln N$, where N is the number of iterations. The upper bound of these plots has slopes equal to $1/(1-q)$ [see Fig. 3(a)]. Its fractal-like structure reflects the presence of long-range temporal correlations at the critical point. The values of q so obtained are plotted in Fig. 4 against the numerical values $1/\alpha_{\min} - 1/\alpha_{\max}$ and corroborate the relation predicted from scaling arguments.

We also computed the multifractal spectrum $f(\alpha)$ and the sensitivity function $\xi(t)$ for the following two-parameter map $x_{t+1} = d\cos(\pi|x_t - 1/2|^{z/2})$ $(1 < z < \infty; 0 < d < \infty; -d \le x_t \le d)$. This map also display a period-doubling route to chaos (z is the inflexion of the map at the vicinity of the extremal point $\bar{x} = 1/2$). A typical onset to chaos is found to be at $d_c(z=2) = 0.865579\ldots$. Our numerical results confirm, as expected, that this map has, for fixed z, the same $f(\alpha)$ and q of the logisticlike map, i.e., both maps belong to the same universality class.

As a final illustration, we computed $\xi(t)$ for the circle map which is an iterative mapping of one point on a circle to another. This map also exhibits a transition to chaos via quasiperiodic trajectories but with a topology distinct from the one of logistic maps. It describes dynamical systems possessing a natural frequency ω_1 and driven by an external frequency ω_2 ($\Omega = \omega_1/\omega_2$ is called the *bare* winding number) and belongs to the same universality class of the forced Rayleigh-Bénard convection [18]. These systems tend to mode lock at a frequency ω_1^* (the ratio between the response frequency and the driving frequency $\omega^* = \omega_1^*/\omega_2$ is usually called the *dressed* winding number). The standard one-dimensional version of the circle map reads $\theta_{t+1} = \theta_t + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_t)$ $(0 < \Omega < 1; 0 < K < \infty; 0 < \theta_t < 1)$. $K = 1$ is the onset

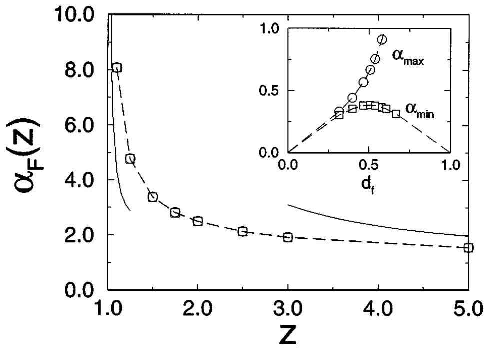

FIG. 2. The logisticlike map values of $\alpha_F(z)$ (numerically obtained from both $\alpha_{\min}$ and $\alpha_{\max}$). The solid lines represent known analytical expressions for the asymptotic behaviors $\{[\alpha_F(z)]^z \to 1/0.033381\ldots$ as $z \to \infty$; and $\alpha_F(z) \sim -1/[(z-1)\ln(z-1)]$ as $z \to 1$ [17]$\}$. Inset: $\alpha_{\min}$ and $\alpha_{\max}$ versus d_f. Dashed lines are guides to the eyes.

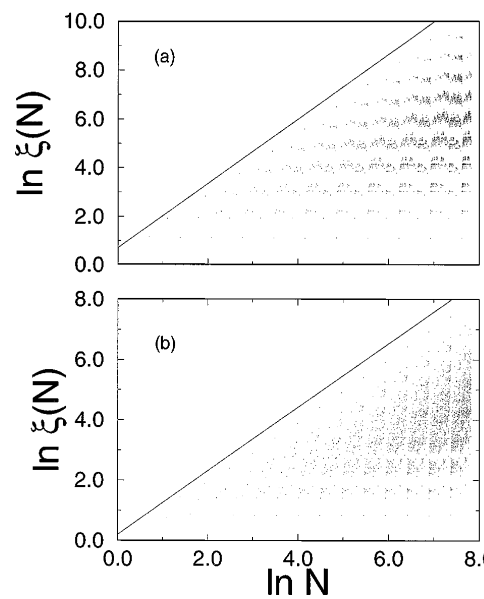

FIG. 3. $\ln\xi(N)$ versus $\ln N$. (a) standard logistic map ($z = 2$). The solid line represents the theoretically predicted slope for the upper bounds $1/(1-q) = \ln\alpha_F(2)/\ln 2 = 1.3236\ldots$ $\{\alpha_F(2) = 2.5029\ldots$ [16]$\}$, hence $q = 0.2445\ldots$. (b) circle map at $K = 1$ and $\omega^* = (\sqrt{5}-1)/2$ $(\Omega = 0.606661\ldots)$. The solid line represents the theoretically predicted slope for the upper bounds $1/(1-q) = 2\ln\alpha_F/\ln\omega^* = 1.0534\ldots$ ($\alpha_F = 1.2885\ldots$ [19]), hence $q = 0.0507\ldots$.

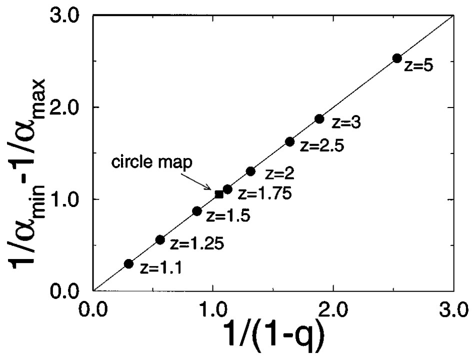

FIG. 4. $1/\alpha_{\min} - 1/\alpha_{\max}$ versus $1/(1-q)$ for the generalized logistic map (circles) and for the circle map (square). The straight line represents the scaling prediction.

value above which chaotic orbits exist (for $K < 1$ the orbits are always periodic). A well-studied transition takes place at $K = 1$ and the dressed winding number equal to the golden mean $\omega^* = (\sqrt{5} - 1)/2 = 0.61803\ldots$, which corresponds to $\Omega = 0.606661\ldots$. With these parameters, the map has a cubic inflexion ($z = 3$) near its extremal point $\bar{\theta} = 0$, and its universal scaling factor is found to be $\alpha_F = 1.2885\ldots$ [19] ($b = 1/\omega^*$ is the natural scale for the partitions). From Eq. (7), the predicted value for the entropic index is $q = 0.0507\ldots$. This value also agrees with the numerical estimation based on the sensitivity to initial conditions [see Fig. 3(b)].

In conclusion, we have shown how the proper nonextensive statistics can be inferred from the scaling properties of the dynamical attractor at the onset of chaos in one-dimensional dissipative maps. The relation between the entropic index q of generalized statistics and the multifractal singularity spectrum of the dynamical attractor was derived using quite general scaling arguments applied to the most concentrated and rarefied regions of the attractor. The proposed relation is therefore expected to hold for higher-dimensional dissipative systems and to provide a close relationship between the nonextensive statistics formalism and the self-organized critical states of large driven dynamical systems. Analogous connections might exist for Hamiltonian systems with long-range interactions.

One of us (C. T.) acknowledges fruitful discussions with J.-P. Eckmann, I. Procaccia, E. M. F. Curado, and C. Anteneodo. This work was partially supported by CNPq, FINEP, and CAPES (Brazilian agencies).

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