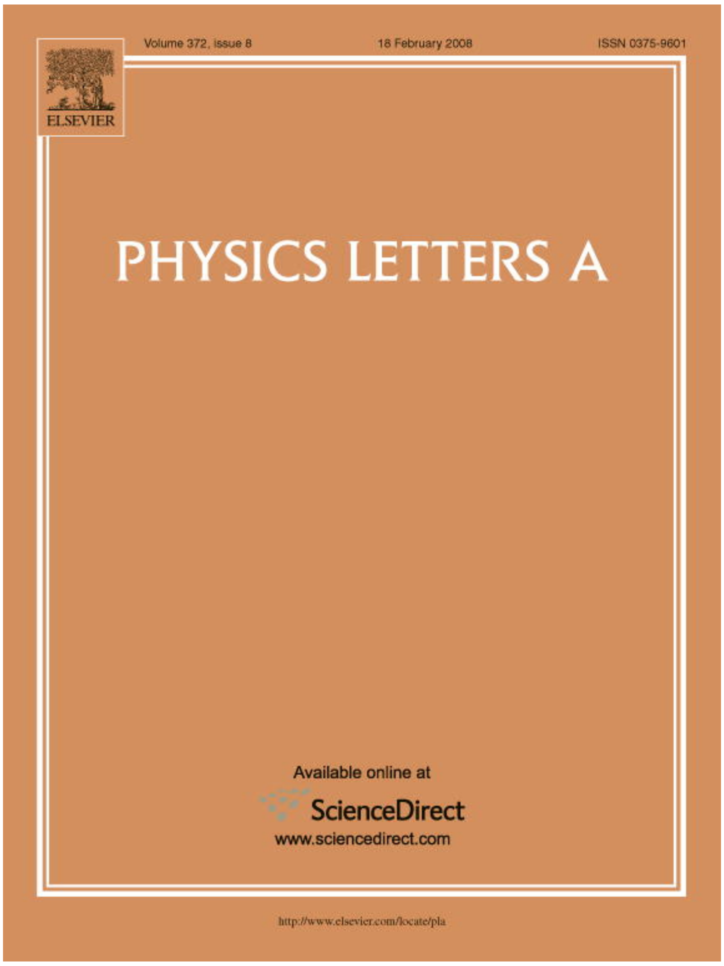
Volume 372, issue 8
18 February 2008
ISSN 0375-9601
ELSEVIER
PHYSICS LETTERS A
Available online at
ScienceDirect
www.sciencedirect.com
http://www.elsevier.com/locate/pla

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Physics Letters A 372 (2008) 1236–1239
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Computing the non-linear anomalous diffusion equation from first principles

Miguel A. Fuentes [a,b,*], Manuel O. Cáceres [b,c]

[a] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA
[b] Centro Atómico Bariloche, Instituto Balseiro and CONICET, 8400 Bariloche, Argentina
[c] The Abdus Salam ICTP, 34014 Trieste, Italy

Received 8 June 2007; received in revised form 3 September 2007; accepted 10 September 2007

Available online 14 September 2007

Communicated by A.R. Bishop

Abstract

We investigate asymptotically the occurrence of anomalous diffusion and its associated family of statistical evolution equations. Starting from a non-Markovian process *à la* Langevin we show that the mean probability distribution of the displacement of a particle follows a generalized non-linear Fokker–Planck equation. Thus we show that the anomalous behavior can be linked to a fast fluctuation process with memory from a microscopic dynamics level, and slow fluctuations of the dissipative variable. The general results can be applied to a wide range of physical systems that present a departure from the Brownian regime.

PACS: 05.70.Ln; 05.30.Pr

Considerable interest and effort has recently been applied to the analysis of anomalous behavior in collective motion. Studies in this area go from turbulence [1], granular matters [2] and economic processes [3] to social behavior [4]. An example of the signature of such processes is the well known anomalous diffusion behavior, where the second moment $\langle x(t)^2\rangle \propto t^\alpha$, with $\alpha \neq 1$, is the archetypal quantity of analysis [5]. This communication aims to introduce some insight into the microscopic foundation *à la Langevin* of such ubiquitous behaviors, passing from the equation of motion of a single particle (microscopic dynamic) to the effective statistical properties of the ensemble system (macroscopic laws).

One intriguing aspect in the description of complex systems is the existence of a non-linear Fokker–Planck equation (NLFP)

$$\frac{\partial P}{\partial t} \propto \frac{\partial^2 P^q}{\partial x^2}, \qquad q > 0, \tag{1}$$

which combines aspects that make it very appropriate to treat, at this point phenomenologically, different fields where anomalous behaviors are relevant (like anomalous transport and long tail probability distributions among others). In fact, this NLFP equation has been applied to disordered systems and porous media [6], where the underlying processes present characteristics of self-similarity, scaling laws, etc., as well as to non-extensive statistical mechanics (see [7] and references therein).

Such equation can be derived, in the case of porous media, combining the continuity equation with two *empirical* relations: Darcy's law and a state equation for polytropic gases (or fluids) $p \propto \rho^\nu$; where p is the pressure and ρ the density [8] of these systems. In non-extensive statistical mechanics theory, for the general case, the NLFP equation has been derived employing self consistent approaches [7,9], using Langevin equations which are *themselves* functions of the probability, that is $\dot{x}(t) = \mathcal{F}[x, P(x,t), \eta(t)]$, where $\eta(t)$ is a white noise. Then, taking into account an appropriated stochastic calculus (Ito, Stratonovich, etc.), it is possible to arrive at the NLFP equation written in Eq. (1).

It is worth stressing that this non-linear evolution equation has been extensively applied to the formalism of non-extensive statistical mechanics, where applications in various scientific fields have been reported, including: long range interaction [10], multifractality [11], behavior at the edge of chaos [12], and others (see [7] and references therein). It

* Corresponding author.
E-mail address: fuentesm@santafe.edu (M.A. Fuentes).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2007.09.020

is therefore a very important task to give an explanation of such non-linear evolution for the probability distribution, which turns out to be of relevance in many complex systems studies.

The above mentioned methodology for deducing the NLFP equation shows an interesting gap in understanding the fundamental underlying processes that make possible the nonlinearity in the evolution equation of this probabilistic function. None of the previous approaches give a clear answer to the problem, apart from showing that those processes can present memory effects at a microscopic level, which may be an important ingredient for the emergence of this particular non-linear (anomalous) evolution.

We will proceed as follows. Starting from one particle dynamics, *à la* Langevin, with memory components, we will be able to infer its asymptotic probability distribution in space and time and the resulting evolution equation. Then, after calculating the average over the slow fluctuations, we will obtain the general expression for the related NLFP equation.

The microscopy dynamics à la Langevin. The presence of a memory kernel in the Langevin equation goes back to works by Kubo, Mori, Nakajima, Zwanzig, et al. (see for example Ref. [13]), but more recently there have been studies showing that a memory kernel is equivalent to the introduction of a fractional differential operator [14]. This has been considered in the Langevin framework, as well as in the Fokker–Planck equation, and makes it possible to describe the anomalous transport process with some degree of accuracy. Let us examine the dynamics of a single particle coupled to a *complex heat bath* with temperature $k_B T$. The equation of motion of such a particle can be written in the form

$$M\ddot{x} + M\int_{0^+}^{t} \gamma(t')\dot{x}(t-t')\,dt' = \xi(t), \tag{2}$$

we have denoted with 0^+ a possible cut-off. Here $\xi(t)$ is a Gaussian long-range correlated noise and $\gamma(t)$ the associated dissipative kernel that can be obtained from the elimination of bath variables [15]. The dissipative kernel $\gamma(t)$ calculated from a microscopic random-matrix model, is

$$Mk_B T\gamma(t) = 2A_0\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)t^{-\alpha}, \quad t > 0, \tag{3}$$

where the exponent α characterizes, in the non-Ohmic regime, the behavior of the spectral density of the bath at low frequencies [16]. The solution we are looking for is subject to the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. It is worth mentioning here that in Ref. [17] we have introduced a functional approach that enables us to solve this kind of linear memory-like Langevin equation in the presence of any arbitrary noise $\xi(t)$, nevertheless, in the present communication we are interested only in a Gaussian noise.

Note that if we rewrite the equation of motion (2) using a fractional derivative, it is possible to see that it differs from the usual Langevin equation by introducing, in the dissipative term, a fractional differential operator of order $\alpha - 1$, for $\alpha \in (0, 2)$, with a coefficient $\gamma_\alpha = \pi A_0/[Mk_B T\sin(\alpha\pi/2)]$ being the dissipation parameter due to complex friction model. As we remarked before, $\xi(t)$ is a Gaussian noise with zero mean and correlation $\langle\xi(t)\xi(0)\rangle = 2A_0\Gamma[\alpha]\cos(\alpha\pi/2)t^{-\alpha}$, with $t > 0$. A_0 is the coupling strength of the particle with the complex bath. Eq. (2) allows us to obtain several results concerning statistical properties of an ensemble of particles subject to slow fluctuations in their dynamic (dissipative) parameter, as we will describe in detail below. In particular we are interested in the description of the position of the particle at a given time, which can be addressed using the marginal probability distribution $P(x,t) = \int P(x,V,t)\,dV$; where $V(t) \equiv \dot{x}(t)$.

Because the noise is Gaussian, the calculation of the two-dimensional joint probability distribution $P(x,V,t)$ is simply done in terms of a few cumulants, then, following [17], we can calculate the marginal probability distribution $P(x,t)$ knowing the second moment of the position, which is given by

$$\langle x^2(t)\rangle = \frac{2kT}{M}t^2 E_{2-\alpha,3}\left(-\gamma_\alpha t^{2-\alpha}\right), \tag{4}$$

where $E_{\mu,\nu}$ is the generalized Mittag–Leffler function [18]. After a transient, the second moment has a clear anomalous behavior, given by [19]

$$\langle x^2(t\to\infty)\rangle \approx \frac{2kT}{M\gamma_\alpha}\frac{t^\alpha}{\Gamma(1+\alpha)} \equiv \frac{t^\alpha}{b}, \tag{5}$$

we have defined $b = M\gamma_\alpha\Gamma(1+\alpha)/(2kT)$. For $\alpha = 1$ this result coincides with the well known diffusive behavior obtained from the asymptotic limit of the Ornstein–Uhlenbeck process [5,20]. The solution for the asymptotic marginal probability distribution is given by

$$P(x,t|b) \equiv P(x,t) = \sqrt{\frac{b}{2\pi t^\alpha}}\exp\left(-\frac{bx^2}{2t^\alpha}\right), \tag{6}$$

where we have denoted explicitly the conditional character of the distribution with the parameter b. This quantity, b, can be seen as a *slow-effective dissipative coefficient* for this anomalous process. We mention here that the evolution equation of such an asymptotic processes is given by a diffusion-like equation, as was also pointed out in [21]

$$\frac{\partial P(x,t|b)}{\partial t} = \frac{\alpha t^{\alpha-1}}{2b}\frac{\partial^2 P(x,t|b)}{\partial x^2}. \tag{7}$$

In fact, if we associate $\alpha = 2H$ with $\alpha \in (0,2)$, $P(x,t|b)$ is the $1 - $ *time* probability distribution of the well known fractional Brownian motion (fBm) process [5,22]. This result shows that the fluctuations at the microscopic level appear as an anomalous dependence in time (anomalous transport), but preserve the Gaussian character of the distribution for fixed times, as expected from the linear Gaussian model (2). For a more general situation see [17].

By introducing a scaling analysis we can calculate the power spectrum from the position of the ensemble of particles. First note that the distribution described by Eq. (6) satisfies the scaling relation

$$P\left(\Lambda^{\alpha/2}x, \Lambda t|b\right) = \Lambda^{-\alpha/2}P(x,t|b), \tag{8}$$

where the prefactor $\Lambda^{-\alpha/2}$ ensures that the probability density is properly normalized, in agreement with the scaling from the evolution Eq. (7). This means that the power spectrum density for the position [5,22,23] will be written as

$$S_x(f) \propto \frac{1}{f^{\alpha+1}}, \quad \alpha \in (0,2). \tag{9}$$

Fluctuations in the dissipative coefficient. In the context of superstatistics [24] the probability distribution $P(x,t|b)$ represents the conditional probability of an event for a given strength b, then, we can obtain a *mean* probability $\mathcal{P}(x,t)$ considering a suitable distribution, $h(b)$, for the dissipative parameter, and making the corresponding average

$$\mathcal{P}(x,t) = \int P(x,t|b)h(b)\,db. \tag{10}$$

Here the support of $\mathcal{P}(x,t)$ is free, i.e. $(-\infty,+\infty)$, and we will leave the discussion to bounded domains, where more complicated $h(b)$ are needed [25], for future communications (we will discuss the difficulties that arise in this case at the end of this communication). The election of the distribution $h(b)$ follows physical insights. We notice that b is a positive quantity, meaning that the distribution must have a positive support. Therefore we will use the gamma distribution that has the properties described before. It is worth mentioning that the gamma distribution has been successfully applied for understanding the slow dynamic aspects of processes described using non-extensive statistical mechanics [24]. In the gamma distribution [26][1] we use the parameters $a = n/2$ and $c = n/(2\beta_0)$ then the mean value is β_0 and its variance gives $\sigma^2 = 2\beta_0^2/n$. After the integration written in Eq. (10) (note that we are interested in averaging a normalized distribution $P(x,t|b)$), we get

$$\mathcal{P}(x,t) = \frac{\Gamma[\frac{(n+1)}{2}]}{\Gamma[\frac{n}{2}]}\sqrt{\frac{\beta_0}{\pi n t^\alpha}}\left(1+\frac{\beta_0}{n}\frac{x^2}{t^\alpha}\right)^{-\frac{n+1}{2}}. \tag{11}$$

We stress that this general relation makes the connection between the fast dynamic of the ensemble of particles and the slow fluctuations of the dissipative parameter b.

Non-linear evolution equation. We can perform the integration over $h(b)$ on Eq. (7) as well. After performing the integration the evolution equation is

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = D(t)\frac{\partial^2 \mathcal{P}(x,t)^{\frac{n-1}{n+1}}}{\partial x^2}, \tag{12}$$

where the parameter $D(t)$ has the following time dependance

$$D(t) \propto t^{\frac{n\alpha}{n-1}-\frac{n+1}{n-1}}. \tag{13}$$

This outcomes shows that for the particular values of

$$n = (1+q)/(1-q), \quad \text{and} \tag{14}$$

$$\alpha = 2/(1+q), \tag{15}$$

the resulting non-linear evolution equation can be written as

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = Q\frac{\partial^2 \mathcal{P}(x,t)^q}{\partial x^2}, \quad q \in (0,1), \tag{16}$$

with

$$Q = \frac{1}{2\beta_0 q}\left[\frac{\pi(1+q)}{\beta_0(1-q)}\right]^{(q-1)/2}\left[\frac{\Gamma[1/(1-q)]}{\Gamma[(1+q)/(2-2q)]}\right]^{1-q}, \tag{17}$$

which, as we mentioned before, appears to be a fundamental NLFP equation that describes densities and distribution functions in many complex systems scenarios, and is also one of the relevant evolution equations for the probability distribution in the non-extensive statistical mechanics theory, as well as in the time series analysis in economic processes [27], for $q < 1$. Note that for the chosen set of parameters, (14) and (15), Q does not depend on time, and also that for large n the normal diffusive regime is obtained.

Some of the implications of the previous result can be seen more clearly in Fig. 1. There, we show how the marginal integration takes into account the slow fluctuation of the parameter b, using the relations given by Eqs. (14)–(15). The mean probability $\mathcal{P}(x,t)$ is bounded between the (two) probabilities $P(x,t|b_\pm)$ for

$$b_\pm = \langle b\rangle \pm \frac{\sigma_\chi^2}{\langle b\rangle}. \tag{18}$$

As expected, for x large enough, the long-tail characteristic of the mean probability $\mathcal{P}(x,t)$ produces a cross over with the Gaussian distribution $P(x,t|b)$; i.e.,

$$\mathcal{P}(x,t) \sim \left(x^2/t^{2/(1+q)}\right)^{1/(1-q)}, \tag{19}$$

for $q < 1$ and $x^2 \gg (\frac{1+q}{1-q})t^{2/(1+q)}/\beta_0$.

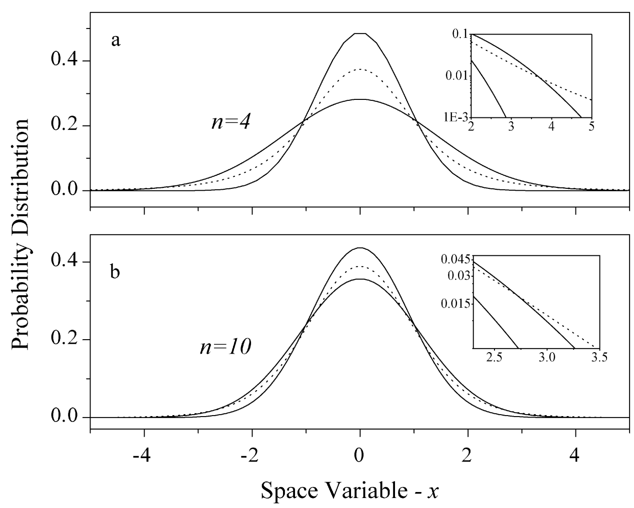

Fig. 1. The $1-time$ probability distributions $P(x,t|b)$, solid line, and $\mathcal{P}(x,t)$, dotted line, vs. the space variable-x. Insets show in log-linear scale the crossover, discussed in the text, between $\mathcal{P}(x,t)$ and $P(x,t|b)$. Time and β_0 are fixed at $\beta_0 = 1$ and $t = 1$ for both cases. (a) $P(x,t|b)$ for $b_\pm = \langle b\rangle \pm \sigma_\chi^2\langle b\rangle^{-1} = \beta_0(1 \pm 2/n)$ with $n = 4$, $\mathcal{P}(x,t)$ for $q = 3/5$, from Eq. (14). (b) Same as before for $n = 10$ and $q = 9/11$.

[1] A gamma distributed random variable y is characterized by the normalized distribution: $f_{a,c}(y) = \frac{c^a}{\Gamma[a]}y^{a-1}\exp(-cy)$, then the moments read $\langle y^m\rangle = a(a+1)\cdots(a+m-1)/c^m$.

For bounded domain, we were unable to find a suitable measure $h(b)$ that satisfies Eq. (10), been $\mathcal{P}(x,t)$ a solution of Eq. (16) for $q > 1$. The main problem is that an important constrain for $h(b)$ is to be independent of x or t, contrary to what is usually used in the literature to approach this case [25] second reference.

Finally, we end with a comment on the associated scaling for the mean distribution $\mathcal{P}(x,t)$, given in Eq. (11) using the relations (14)–(15). In this case $\mathcal{P}(x,t)$ and its evolution equation fulfill the scaling relation

$$\mathcal{P}\big(\Lambda^{1/(q+1)}x, \Lambda t\big) = \Lambda^{-1/(q+1)}\mathcal{P}(x,t), \tag{20}$$

then the corresponding power spectrum density is characterized by

$$\mathcal{S}_x(f) \propto \frac{1}{f^{(q+3)/(q+1)}}, \quad 0 \leqslant q \leqslant 1. \tag{21}$$

This spectrum belongs to a different class of universality when compared with (9). The spectrum (21) can be related to that corresponding to a *generalized* Wiener process where the noise has a long-range correlation, therefore leading to strong non-Markovian behavior [23]. In other words, for $q < 1$ the spectrum of the $1 - time$ distribution $\mathcal{P}(x,t)$ can be mapped to the *persistent* case of the fBm, but not to the *antipersistent* one.

Finals remarks. In summary, we have linked a memory-like Langevin (microscopic) dynamic with the general non-linear Fokker–Planck equation. Our results show that in this framework it is possible to address the $1 - time$ probability distribution and its non-linear evolution equation analytically and, therefore, all the relevant objects associated with them. The solution of this general NLFP equation present anomalous transport and long tail behavior for fixed times. We should mention that this approach is valid not only for the specific example discussed in this communication, but for the complete family of evolution equations that arises using fractional differential operators (or their related dissipative memory kernels).

Acknowledgements

M.A.F. would like to thank Prof. A. Tsuda and Prof. Y. Sato for fruitful discussions related to memory effects in discrete dynamics processes during his stay at Hokkaido University, and CONICET-CNRS Res. 603/07. M.O.C. thanks grants from SECTyP, Univ. Nac. Cuyo, and PIP 2005 CONICET, Argentina. The authors thanks L. Ware, from Santa Fe Institute, for the English revision.

References

[1] A. La Porta, G.A. Voth, A.M. Crawford, J. Alexander, E. Bodenschatz, Nature (London) 409 (2001) 1017.
[2] J.E. Fiscina, M.O. Cáceres, J. Phys.: Condens. Matter 17 (2005) S1237; J.E. Fiscina, M.O. Cáceres, Phys. Rev. Lett. 95 (2005) 108003.
[3] J.P. Bouchaud, M. Potters, Theory of Financial Risks, Cambridge Univ. Press, Cambridge, 2000.
[4] D. Brockmann, L. Hufnagel, T. Geisel, Nature (London) 439 (2006) 4292.
[5] M.O. Cáceres, Estadística de no equilibrio y medios desordenados, Reverté S.A., Barcelona, ISBN 84-291-5031-5, 2003 (in Spanish).
[6] D.G. Aronson, in: A. Dobb, D. Eckmann (Eds.), Nonlinear Diffusion Problems, in: Lectures Notes in Mathematics, vol. 1224, Springer, Berlin, 1986;
G.I. Barenblatt, V.M. Entov, V.M. Ryshik, Theory of Fluid Flows Through Natural Rocks, Kluwer Academic, Dordrecht, 1990;
R.E. Pattle, Quart. J. Mech. Appl. Math. 12 (1959) 407;
L.A. Peletier, The porous media equation, in: H. Amann, N. Bazley, K. Kirchgassner (Eds.), Applications of Nonlinear Analysis in the Physical Science, Pitman Advanced Publishing Program, Boston, 1981.
[7] M. Gell-Mann, C. Tsallis, Editors, Nonextensive Entropy: Interdisciplinary Applications, Oxford Univ. Press, New York, 2004.
[8] T.D. Frank, Nonlinear Fokker–Planck Equations, third ed., Springer, Berlin, 2004.
[9] L. Borland, Phys. Rev. E 57 (1998) 6634.
[10] J. Du, Phys. Lett. A 329 (2004) 262.
[11] T. Arimitsu, N. Arimitsu, J. Phys. A 33 (2000) L235.
[12] F. Baldovin, A. Robledo, Phys. Rev. E 69 (2004) 045202R.
[13] R. Kubo, M. Toda, N. Hashitsume, Statistical Physics II, Nonequilibrium Statistical Mechanics, Springer, Berlin, 1985;
H.J. Kreuzer, Nonequilibrium Thermodynamics and its Statistical Foundations, Clarendon, Oxford, 1981.
[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[15] E. Lutz, Phys. Rev. E 64 (2001) 051106.
[16] E. Lutz, Europhys. Lett. 54 (2001) 293.
[17] A.A. Budini, M.O. Cáceres, J. Phys. A: Math. Gen. 37 (2004) 5959.
[18] R. Hilfer, Editor, Applications of Fractional Calculus in Physics, World Scientific, 2000.
[19] A. Erdelyi, Higher Transcendental Functions, vol. 3, McGraw–Hill, New York, 1955.
[20] C.W. Gardiner, Handbook of Stochastic Methods, Springer, Berlin, 2005.
[21] K.G. Wang, Phys. Rev. A 45 (1992) 833.
[22] B.B. Mandelbrot, J.W. Van Ness, SIAM Rev. 10 (1968) 422.
[23] M.O. Caceres, Phys. Rev. E 60 (1999) 5208;
M.O. Caceres, Brazilian J. Phys. 29 (1999) 125.
[24] C. Beck, Phys. Rev. Lett. 87 (2001) 18;
C. Beck, E.G.D. Cohen, Physica A 322 (2003) 267.
[25] G. Wilk, Z. Wlodarczyk, Phys. Rev. Lett. 84 (2000) 13;
G. Wilk, Z. Wlodarczyk, hep-ph/0004250, unpublished.
[26] W. Feller, in: An Introduction to Probability Theory and its Applications, vol. 2, second ed., John Wiley & Sons, New York, 1971.
[27] S.M.D. Queirós, Quantitative Finance 5 (5) (2005).