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Nonextensive Pesin identity: Exact renormalization group analytical results for the dynamics at the edge of chaos of the logistic map

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We show that the dynamical and entropic properties at the chaos threshold of the logistic map are naturally linked through the nonextensive expressions for the sensitivity to initial conditions and for the entropy. We corroborate analytically, with the use of the Feigenbaum renormalization group transformation, the equality between the generalized Lyapunov coefficient λ_q and the rate of entropy production, K_q, given by the nonextensive statistical mechanics. Our results advocate the validity of the q-generalized Pesin identity at critical points of one-dimensional nonlinear dissipative maps.

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The nonextensive generalization of the Boltzmann-Gibbs (BG) statistical mechanics [1,2] has recently raised much interest and provoked considerable debate [3] as to whether there is firm evidence for its applicability in circumstances where a system is out of the range of validity of the canonical BG theory. Recognition and understanding of the existence of such a limit of validity is a major concern in the development of present-day statistical physics. So, increased attention has been drawn to the examination of physical situations that do not satisfy the customary BG equilibrium conditions, e.g., insufficient randomness and limited or nonuniform motion over pertinent phase space, that result in anomalous dynamical properties [4,5]. Within these several types of physical phenomena and connected model system properties [2], the critical states of one-dimensional nonlinear dissipative maps stand out. It has become apparent [2] that they are *bona fide* examples where the predictions of the nonextensive generalization of the BG statistical mechanics are appropriate. Here we give analytical proof and numerical corroboration of the until now only conjectured [6] q generalization of the Pesin identity between entropy production rate and Lyapunov exponent at the onset of chaos of unimodal maps. Beyond its intrinsic value as a means to study the dynamics of incipient chaotic states, this result has important implications regarding the suitability of the Tsallis entropy in describing a specific physical situation unreachable by BG statistics.

Recently, the predictions of the nonextensive theory have been rigorously proved for the pitchfork and tangent bifurcations and for the edge of chaos of logistic-type maps by means of the analytic renormalization group (RG) derivation [7] of the q-exponential expression,

$$\xi_t = \exp_q(\lambda_q t) \equiv [1-(q-1)\lambda_q t]^{-1/(q-1)} \tag{1}$$

for the sensitivity to initial conditions, ξ_t, containing the entropic index q and the q-generalized Lyapunov coefficient λ_q. Equation (1) has been proposed [6] as the nonextensive counterpart to the usual exponential sensitivity $\xi_t = \exp(\lambda_1 t)$ to initial conditions which prevails when the ordinary Lyapunov coefficient λ_1 is nonvanishing. (The BG exponential form is recovered when $q \to 1$.)

Pioneering work [8,9] on the dynamics at the edge of chaos of the logistic map was directed at the determination of the fluctuation spectrum of the algebraic Lyapunov coefficients λ_q. Here we focus on the entropic properties of trajectories at this state with the idea of investigating the existence of a generalized Pesin identity.

As a significant windfall of our renormalization group (RG) calculations we now know the expressions for λ_q at the mentioned critical states of logisticlike maps [7]. These expressions have been interpreted in terms of the fixed-point map parameters and corroborated numerically via *a priori* calculations [7]. Specifically, for the edge of chaos μ_∞ of the logistic map λ_q (and q) are simply given by $\lambda_q = \ln \alpha / \ln 2$ (and $q = 1 - \ln 2 / \ln \alpha$), where α is the Feigenbaum's universal constant that measures the power-law period-doubling spreading of iterate positions. Having reached this level of knowledge on ξ_t and λ_q it is only natural to enquire about its relationship with the entropic properties of trajectories at μ_∞. The Pesin formula that relates the Kolmogorov-Sinai (KS) entropy $\mathcal{K}_1$ (described below) and the Lyapunov coefficients of nonlinear maps has become an extremely useful tool for the quantitative analysis of the dynamics of chaotic states. This formula embodies the all-important connection between the loss of information measured by $\mathcal{K}_1$ and the Lyapunov coefficients $\lambda_1^{(l)}$ for chaotic states [10,11]. The general inequality $\mathcal{K}_1 \leqslant \Sigma \lambda_1^{(l)}$ where the sum is over the $\lambda_1^{(l)} > 0$ reduces for one-dimensional systems to the Pesin identity $\mathcal{K}_1 = \lambda_1$, $\lambda_1 > 0$.

So, as a starting point we consider the q-generalized rate of entropy production K_q, defined via $K_q t = S_q(t) - S_q(0)$, t large, where

$$S_q \equiv \sum_i p_i \ln_q\left(\frac{1}{p_i}\right) = \frac{1 - \sum_i^W p_i^q}{q-1} \tag{2}$$

is the Tsallis entropy and where $\ln_q y \equiv (y^{1-q}-1)/(1-q)$ is the inverse of $\exp_q(y)$. In the limit $q \to 1$ K_q becomes K_1

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$\equiv t^{-1}[S_1(t)-S_1(0)]$ where $S_1(t)=-\Sigma_{i=1}^{W} p_i(t)\ln p_i(t)$. In Eq. (2) $p_i(t)$ is the probability distribution obtained from the relative frequencies with which the positions of an ensemble of trajectories occur within cells $i=1,\ldots,W$ at iteration time t. The initial conditions for these trajectories have a prescribed distribution $p_i(0)$ and the phase space into which the map is defined is partitioned into a large number W of disjoint cells of sizes l_i. As a difference from K_1 the KS entropy has a more elaborate definition since it considers the entire trajectories from their initial positions to the time limit $t\to\infty$ [11]. The relationship between $\mathcal{K}_1$ and K_1 has been investigated for several chaotic maps [12] and it has been established that the equality $\mathcal{K}_1=K_1$ occurs during an intermediate stage in the evolution of the entropy $S_1(t)$, after an initial transient dependent on the initial distribution and before an asymptotic approach to a constant equilibrium value. The q-generalized KS entropy $\mathcal{K}_q$ is defined in the same manner as $\mathcal{K}_1$ but with the use of Eq. (2). Here we look into the analogous intermediate regime in which $\mathcal{K}_q=K_q$. As we shall see below it turns out to be sufficient to evaluate the rate K_q for uniform initial distributions defined in a partition of equal-sized cells to establish the validity of the conjectured [6] form $K_q=\lambda_q$ of the Pesin identity.

Next we recall that the logistic map $f_\mu(x)=1-\mu x^2$, $-1\leqslant x\leqslant 1$, exhibits several types of infinite sequences of critical points (with $\lambda_1=0$) as the control parameter μ varies across the interval $0\leqslant\mu\leqslant 2$. One such important sequence corresponds to the pitchfork bifurcations [10]. The accumulation point of the pitchfork bifurcations is the Feigenbaum attractor that marks the dividing state between periodic and chaotic orbits, at $\mu_\infty=1.401\,15\ldots$. A measure of the amplitudes of the periodic orbits is defined by the diameters d_n $(n=0,1,\ldots)$ of the "bifurcation forks" at the "superstable" periodic orbits of lengths 2^n that contain the point $x=0$. These superstable orbits occur at $\overline{\mu}_n<\mu_\infty$ and approach μ_∞ as $\overline{\mu}_n-\mu_\infty\sim\delta^{-n}$ (n large) where $\delta=0.466\,92\ldots$ is one of the two Feigenbaum's universal constants. The diameter $d_n\equiv f_{\overline{\mu}_n}^{(2^{n-1})}(0)$ is the iterate position closest to $x=0$ in such 2^n cycle. For large n these distances have constant ratios $d_n/d_{n+1}=-\alpha$, where $\alpha=2.502\,90\ldots$ is the second of the Feigenbaum's constants [10]. For clarity we use only the absolute values of positions, so below d_n means $|d_n|$.

The main points in the following analysis are as follows.

(1) We determine the evolution of *all* orbits at μ_∞, i.e., those with initial conditions x_{in} belonging to the attractor, to the repeller, and to all other positions.

(2) We obtain ξ_t for any x_{in} and find the remarkable property that Eq. (1) holds in general with the same fixed values for q and λ_q up to a time $T=2^N$ where $N\simeq-\ln x_{in}/\ln\alpha$.

(3) We observe that the position-independent form found for λ_q implies that ensembles of trajectories expand in such a way that a uniform distribution of initial conditions remains uniform for all later times $t\leqslant T$, where T marks the crossover to an asymptotic regime.

(4) As a consequence of this we establish the identity of the rate of entropy production K_q with λ_q. We corroborate numerically all our findings.

In Fig. 1 we show the absolute values of the positions $x_\tau\equiv|f_{\mu_\infty}^{(\tau)}(x_{in})|$ of two trajectories of the logistic map at μ_∞ in

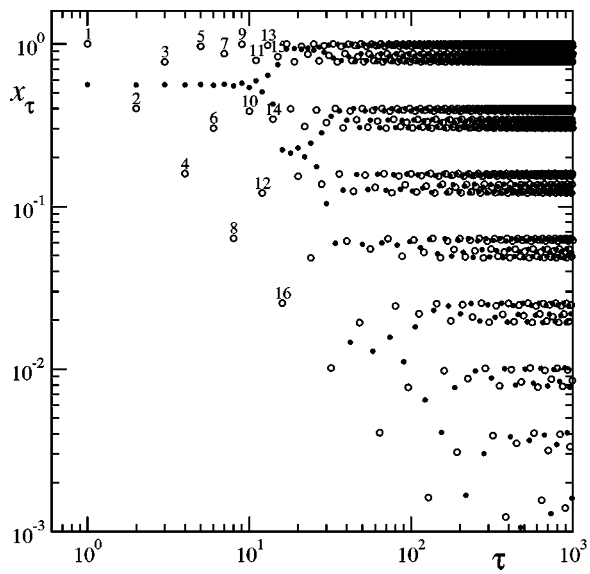

FIG. 1. Absolute value of two trajectories at μ_∞ in logarithmic scales. Empty circles correspond to $x_{in}=0$ (the numbers label time $\tau=1,\ldots,16$). Small dots correspond to $x_{in}\simeq 0.560\,23\ldots$, close to a repeller, the unstable solution of $x=1-\mu_\infty x^2$.

logarithmic scales. One corresponds to $x_{in}=0$, and the other one to $x_{in}\simeq 0.560\,23\ldots$, close to a repeller, the unstable solution of $x=1-\mu_\infty x^2$. The first trajectory maps out the attractor while the second exhibits a long-lived transient stretch as it falls into it. In Ref. [7] it was shown that a distinct fraction of the positions of the trajectory with $x_{in}=0$ consists of subsequences generated by the time subsequences $\tau_k=2^n+2^{n-k}$, with $k=0,1,\ldots$ and $n\geqslant k$, and each of these exhibits the same power-law decay. The main subsequence ($k=0$) can be expressed (via a time variable shift $t_0=\tau_0-1$) as the q-exponential $x_{t_0}=\exp_Q(\Lambda_Q t_0)$ with $Q=1+\ln 2/\ln\alpha$ and $\Lambda_Q=-\ln\alpha/\ln 2$. The positions in the subsequences can be obtained from those belonging to the supercycles at $\overline{\mu}_n<\mu_\infty$. In particular, the subsequence for $k=0$ was identified as the diameter sequence $x_{2^n}=d_n=\alpha^{-n}$. This property was shown to imply Eq. (1) with $q=1-\ln 2/\ln\alpha$ and $\lambda_q=\ln\alpha/\ln 2$. Notice that $q=2-Q$ as $\exp_q(y)=1/\exp_Q(-y)$.

We can extend considerably the above results. First, we note that the *entire* attractor can be decomposed into position subsequences associated with the same power-law decay, and that *all* subsequences are generated by the time subsequences $\tau_k=(2k+1)2^{n-k}$. (The first position at $\tau=2k+1$ of each of the first eight subsequences can be identified among those labeled in Fig. 1.) We make use of this time-position classification to point out that the positions x_{τ_k} of *all trajectories* with $x_{in}<\alpha^{-m}$, $m=n-k$, are given by

$$x_{\tau_k}\equiv|g^{(\tau_k)}(x_{in})|\simeq\frac{g_k(0)}{\alpha^m}+\frac{g_k''(0)}{2\alpha^{-m}}x_{in}^2, \qquad (3)$$

where we have neglected terms of $O(\alpha^{3m}x_{in}^4)$ and where $g_k\equiv g^{(2k+1)}(x)$ is the $(2k+1)$th composition of the fixed-point map $g(x)$. Both $g(x)$ and $g_k(x)$ are solutions of the RG doubling transformation consisting of functional composition and rescaling, $\mathbf{R}f(x) \equiv \alpha f(f(x/\alpha))$. Equation (3) is obtained by keeping the first two terms in the power-series expansion of $g_k(x)$ [10] followed by use of $g_k(x) = \alpha^m g_k^{(2^m)}(x/\alpha^m)$ and the change of variable $x_{in} \equiv \alpha^{-m}x$. Considering a pair of initial conditions y_{in} and x_{in} in Eq. (3) yields

$$x_{\tau_k}(y_{in}) - x_{\tau_k}(x_{in}) = [x_{2k+1}(y_{in}) - x_{2k+1}(x_{in})]\alpha^m. \qquad (4)$$

For each subsequence k, the sensitivity ξ_{t_k}, defined as

$$\xi_{t_k} \equiv \lim_{|y_{in}-x_{in}|\to 0} \frac{|x_{t_k}(y_{in}) - x_{t_k}(x_{in})|}{|x_{t_k=0}(y_{in}) - x_{t_k=0}(x_{in})|}, \qquad (5)$$

can be written, with use of the shifted time variable $t_k \equiv \tau_k - 2k - 1$ $(n \geqslant k)$, and observing that $\alpha^m = [1 + t_k/(2k+1)]^{\ln \alpha/\ln 2}$, as the q exponential:

$$\xi_{t_k} = \exp_q[\lambda_q^{(k)} t_k], \qquad (6)$$

where $q = 1 - \ln 2/\ln \alpha$ and $\lambda_q^{(k)} = \ln \alpha/[(2k+1)\ln 2]$. The crossover time $T = 2^N$ is determined from the condition $x_{in} < \alpha^{-N}$ in Eq. (3).

Let us consider next an ensemble of $\mathcal{N}$ trajectories with initial positions x_{in} uniformly distributed along the interval $[1-l, 1]$, for transparency with $1 - l \leqslant g^{(3)}(0)$. An arbitrary partition of $[1-l, 1]$ is made with a certain number I of nonintersecting intervals of lengths l_i, $i = 1, 2, \ldots, I$, with $l = \Sigma_i\, l_i$. For l sufficiently small, under $\tau_k = (2k+1)2^{n-k}$ iterations the lengths l_i transform, according to Eq. (4), as $l_i^{(\tau_k)} = \alpha^m l_i$. Since we also have $l^{(\tau_k)} = \alpha^m l$, we observe that the interval ratios remain constant, that is, $l_i/l = l_i^{(\tau_k)}/l^{(\tau_k)}$. Thus, the initial number of trajectories within each interval $\mathcal{N} l_i/l$ remains fixed for all times $\tau < T$, with the consequence that the original distribution is uniform for all times $\tau < T$.

We can now calculate the rate of entropy production. This is more easily done with the use of a partition of W equal-sized cells of length l. Figure 2 provides a striking corroboration of the time constancy of uniformity. This figure shows in logarithmic scales the evolution of a distribution $p_i(\tau)$ of positions of an ensemble of trajectories at μ_∞ beginning from a uniform distribution $p_i(1)$ of initial positions contained within a single cell of size l adjacent to $x = 1$. If we denote by W_{t_k} the number of cells that the ensemble occupies at the shifted time t_k and by Δx_{t_k} the total length of the interval these adjacent cells form, we have $W_{t_k} = \Delta x_{t_k}/l$ and in the limit $l \to 0$ [since $W_{t_k} = (\Delta x_{t_k}/\Delta x_{t_k=0})(\Delta x_{t_k=0}/l)$] we obtain the remarkably simple result $W_{t_k} = \xi_{t_k}$. As the distribution is uniform, and recalling Eq. (6) for ξ_{t_k}, the entropy is given by $S_q(t_k) = \ln_q W_{t_k} = \lambda_q^{(k)} t_k$, while

$$K_q^{(k)} = \lambda_q^{(k)} \qquad (7)$$

as $W_{t_k=0} = 1$. Equation (7) is our main result. The numerical results shown in Fig. 3 substantiate and bring to light in a dramatic manner the validity of the q-generalized Pesin identity at μ_∞.

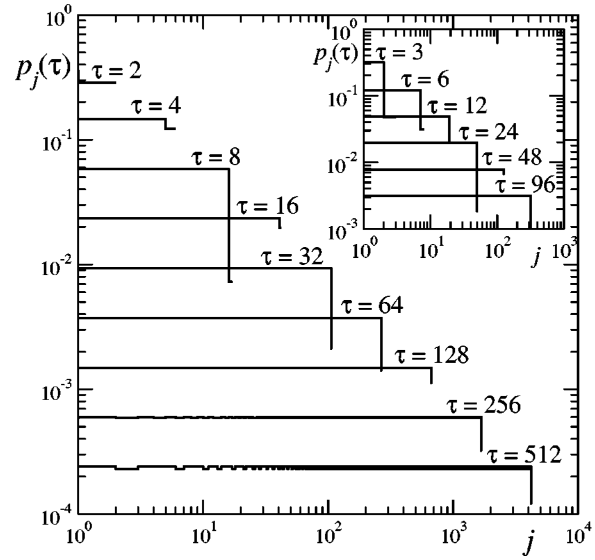

FIG. 2. Time evolution, in logarithmic scales, of a distribution $p_j(\tau)$ of trajectories at μ_∞. Initial positions are contained within a cell adjacent to $x = 1$ and j counts the consecutive location of the occupied cells at time τ. Iteration time is shown for the first two subsequences $(k = 0, 1)$.

An interesting observation about the structure of the nonextensive formalism is that the equiprobability entropy expression $\ln_q W_t$ can be obtained not only from S_q in Eq. (2) but also from

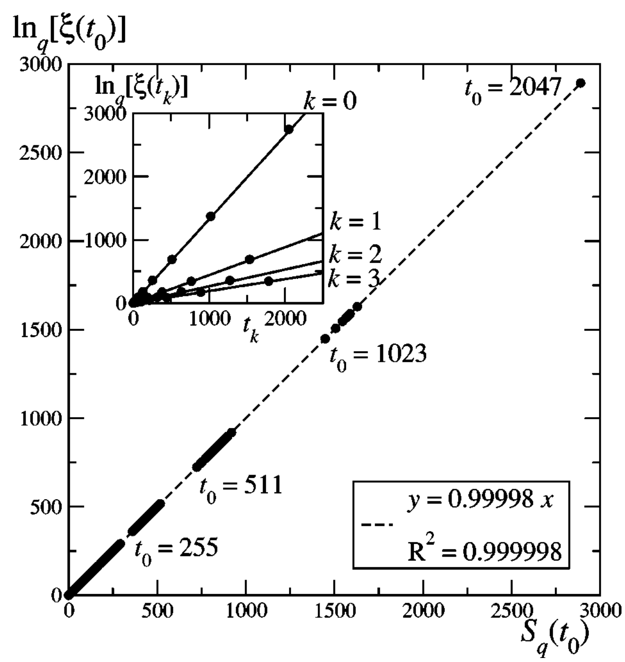

FIG. 3. Numerical corroboration (full circles) of the generalized Pesin identity $K_q^{(k)} = \lambda_q^{(k)}$ at μ_∞. On the vertical axis we plot the q logarithm of ξ_{t_k} (equal to $\lambda_q^{(k)} t$) and in the horizontal axis S_q (equal to $K_q^{(k)} t$). In both cases $q = 1 - \ln 2/\ln \alpha = 0.2445\ldots$. The dashed line is a linear fit. In the inset the full lines are from the analytical result, Eq. (6).

$$S_Q^{\dagger} \equiv -\sum_{i=1}^{W} p_i \ln_Q(p_i), \qquad (8)$$

where $S_Q^{\dagger}=S_{2-Q}=S_q$. The inverse property of the q exponential reads $\ln_q y=-\ln_{2-q}(1/y)$ for the q logarithm and as pointed out introduces a pair of conjugate indices $Q=2-q$ with the consequence that while some theoretical features are equally expressed by both S_q and $S_Q^{\dagger}$ some others appear only via the use of either S_q or $S_Q^{\dagger}$. For instance, the canonical ensemble maximization of $S_Q^{\dagger}$ with the customary constraints $\Sigma_{i=1}^{W} p_i=1$ and $\Sigma_{i=1}^{W} p_i\epsilon_i=U$, where ϵ_i and U are configurational and average energies, respectively, leads to a Q-exponential weight (with $Q>1$ when $q<1$). On the other hand the partition function is obtained via the optimization of S_q. The mutual equations (2) and (8) elegantly generalize the BG entropy.

We summarize our arguments and findings. Critical states with vanishing λ_1 in dissipative one-dimensional nonlinear maps display power law ξ_t. It is natural to expect this to imply a corresponding power-law rate of entropy production linked to the dynamics of ensembles of trajectories. A connection between these two properties suggests an extension of the Pesin identity $\mathcal{K}_1=\lambda_1$, $\lambda_1>0$ that incorporates the case $\lambda_1=0$. But, interestingly, to study this situation formally one is required to develop a theory beyond the usual BG scheme for chaotic states that supplies generalizations for both the KS entropy and the Lyapunov exponent. One known source is the nonextensive statistics constructed around the Tsallis entropy S_q as this offers specific and practicable expressions for these quantities. To make a meaningful analysis of this problem it is indispensable to carry out an explicit *a priori* determination of all quantities involved and here we have proved that this is indeed the case for a specific but prototypical example, the onset of chaos of the logistic map. We have shown [7] that the Feigenbaum RG method, from which the static fixed-point solution $g(x)$ was originally obtained, is also capable of delivering dynamical properties, most visibly the sensitivity ξ_t. It is important to stress that the derivation of ξ_t does not use in any way the nonextensive formalism and for this reason it constitutes an independent corroboration of the expression for ξ_t suggested by this theory. With an analytical expression for λ_q in hand a parallel expression for the rate of energy production K_q was here obtained from two ingredients. (i) a distribution function $p_i(t)$ of positions for an ensemble of trajectories and (ii) the Tsallis entropy S_q. Our main result, the generalized identity $K_q=\lambda_q$, necessitates that the equiprobability entropy has the precise analytical form $\ln_q W_t$ (with $q=1-\ln 2/\ln \alpha$) and to this extent distinguishes the Tsallis expression S_q from other alternatives, including the BG S_1.

We have shown that the Pesin identity holds rigorously, albeit in a generalized form, for incipient chaotic states. Because the entropic index q (as is the case of λ_q and its identity K_q) is obtainable in terms of the Feigenbaum's α we are able to address the much-asked question regarding the manner in which the index q and related quantities are determined in a physical application. The generic chaotic state is that associated to $\lambda_1>0$, but it is evident that the critical state with $\lambda_1=0$ carries with it completely different physics. The analysis was specifically carried out for the Feigenbaum attractor of the logistic map but our findings clearly have a universal validity for the entire class of unimodal maps and its generalization to other degrees of nonlinearity. In a more general context our results indicate a limit of validity to the BG theory based on S_1 and the appropriateness of the nonextensive S_q for this kind of critical dynamic states.

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